The Culture of Quaternions

The Phoenix Bird of Mathematics

Herb Klitzner

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http://quaternions.klitzner.org

The Phoenix Bird

CONTENTS

Introduction
The Word “Quaternion”

- The English word quaternion comes from a Latin word *quaterni* which means grouping things **“four by four.”**
- A passage in the New Testament (Acts 12:4) refers to a **Roman Army detachment of four quaternions** – 16 soldiers divided into groups of four, who take turns guarding Peter after his arrest by Herod. So a quaternion was a **squad of four soldiers**.
- **In poetry**, a quaternion is a **poem using a poetry style** in which the theme is divided into four parts. Each part explores the complementary natures of the theme or subject. [*Adapted from Wikipedia*]
- In mathematics, quaternions **are generated from four fundamental elements (1, i, j, k).**
- Each of these four fundamental elements is associated with a unique dimension. **So math quaternions are, by nature, a 4D system.**

Introduction
The Arc of Dazzling Success and Near-Total Obscurity

Quaternions were created in 1843 by William Hamilton.

Today, few contemporary scientists are familiar with, or have even heard the word, **quaternion**. (Mathematical physics is an exception.) And yet --

- During the 19th Century quaternions became very popular in Great Britain and in many universities in the U.S. (Example: Harvard)
- **Maxwell** advocated the selective use of quaternions as an aid to science thinking about relationships, but not necessarily as a calculating tool.

In the 20th Century (after 1910), quaternions were essentially discarded by most of the math profession when the tools of vector analysis and matrix algebra became sufficiently developed and popularized. A small minority of researchers continued to see their value, especially for modeling, among them developmental psychologist Jean Piaget around 1915.

- **Ironically, the basic ideas of vector analysis were derived from Hamilton's quaternions.**
- Echoing the **Phoenix Bird (see above)** and its specialized proliferation in the last 20-25 years, quaternions have been discovered by a new generation of cutting-edge engineers and scientists in many fields.

Introduction
Surprising Resurgence and Emerging Use in Biology & Neuroscience

In the 20th Century, especially in the closing decades, quaternions have been applied successfully to every level of nature:

- from aerospace navigation to quantum physics spin
- from DNA string analysis to explaining child development of logic.

Quaternion systems do the following well:

- perform rotations
- determine orientation
- shift viewpoint of perception
- filter information
- provide process control.

Introduction – The Importance of 4D Processing in Cognition and Music

MY RESEARCH INTEREST:

Neuroscience: My own conjecture is that quaternion processes are related to the 3D multisensory spatial synthesis of the parietal lobe and to the thalamus, which is a connecting, controlling, and re-imaging structure of the brain.

Four-dimensional models: I am particularly interested in the extension of certain of these 3D cognitive imaging process models to 4D. I see music cognition as a good window into this question, including the perception of melody as possibly 4D.

Introduction – The Importance of 4D Processing in Cognition and Music

Respected cognitive researchers who endorse the 4D nature of selected cognitive processes:
Note: I have corresponded with all of them. Several are quaternion advocates. Several conclude that melody is 4D in nature.

- Arnold Trehub
- Mike Mair
- Terry Marks-Tarlow
- Ben Goertzel
- Mike Ambinder

Quotations

- Quaternions came from Hamilton after his really good work had been done, and though beautifully ingenious, **have been an unmixed evil** to those who touched them in any way, including Clerk Maxwell.

  (*Lord Kelvin, 1892, Letter to Heyward). Quoted by Simon Altmann in Rotations, Quaternions, Double Groups*).

- **"Our results testify that living matter possesses a profound algebraic essence. They show new promising ways to develop algebraic biology."**
  (*Petoukhov, 2012, from his DNA research using quaternion and octonion methods, in* The genetic code, 8-dimensional hypercomplex numbers and dyadic shifts)

Quotations

“An interest [in] quaternionic numbers essentially increased in last two decades when a new generation of theoreticians started feeling in quaternions deep potential yet undiscovered.“

A.P. Yefremov (2005)

“Quaternions…became a standard topic in higher analysis, and today, they are in use in computer graphics, control theory, signal processing [including filtering], orbital mechanics, etc., mainly for representing **rotations and orientations in 3-space**.”

Waldvogel, Jorg (2008)

History

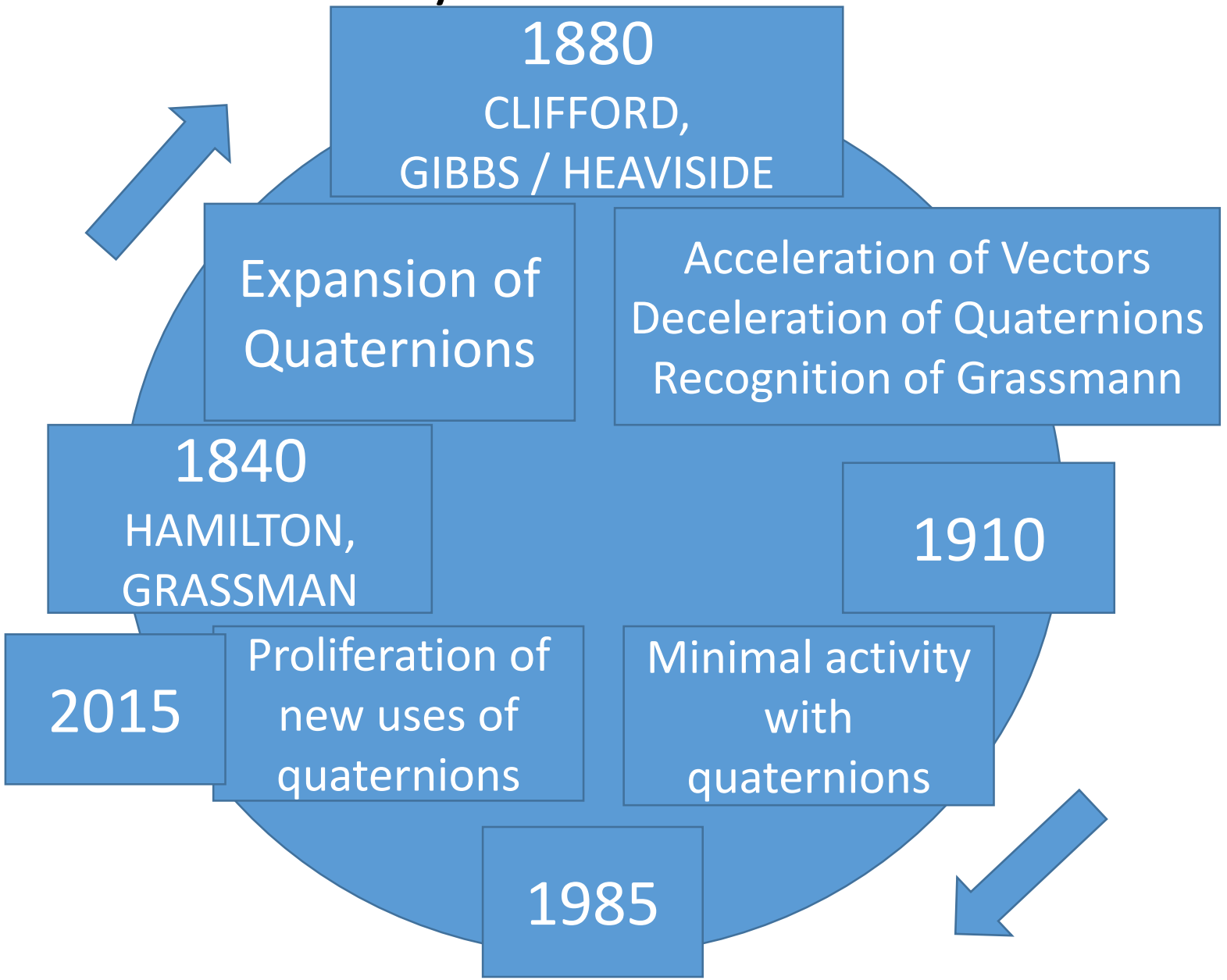
Phoenix Cycle (diagram)

Ada Lovelace, Clerk Maxwell
(Lovelace was the collaborator with Babbage on a proto-computer)

Benjamin Peirce, Jean Piaget

Vector Evolution and Controversies

History Overview – Quaternions vs Vectors



TIME CIRCLE
1840-2015

History Overview -- Personalities

Per-iod	Era	Personalities
1	Mid-19th C.	Wm. Hamilton (1843), Robt. Graves (1843), Hermann Grassmann (1832, 1840, 1844), Olinde Rodrigues (1840) Ada Lovelace (1843)
2	2nd half 19th C.	Benjamin Peirce (1870), Charles Sanders Peirce (1882), Peter Tait (1867), Clerk Maxwell (1873), (Josiah) Willard Gibbs (1880-1884), Oliver Heaviside (1893), Wm. Clifford (1879), Felix Klein
3	1st Half 20th C.	Jean Piaget (1915), Wolfgang Pauli (1927), Paul Dirac (1930, 1931), E.T. Whittaker (1904, 1943), L. L. Whyte (1954), Nicolas Tesla, E.B. Wilson (1901)
4	2nd half 20th C.	David Hestenes (1966, 1987), Ken Shoemake (1985), Karl Pribram (1986), John Baez (2001), NASA, Ben Goertzel (2007)
	Historians of Math	Michael Crowe (1967), Daniel Cohen (2007), Simon Altmann (1986)
	Philosophers and Educators of Math	Ronald Anderson (1992), Andrew Hanson (2006), Jack Kuipers (1999), Doug Sweetser (2014, www.quaternions.com)

Ada Lovelace (1843)

The square route of negative numbers captivated her. A negative number, when squared, equals a positive one. What number when squared could equal a negative one? A completely different kind of number: an imaginary number. Its attributes could be explained by plotting a graph showing the method of combination of real with imaginary numbers with an axis representing the imaginary component of the complex number and an axis representing the real numbers:

> This created a new form of 2-dimensional geometry (a geometry that 160 years later would be shown to have as one of its distinctive shapes that icon of chaos theory, the Mandelbaum Set).
>
> Ada's response to this was to ask what sounded like a simple, technical question but which was in fact a deeply profound one: could you have a third set of numbers in addition to real and imaginary ones which would yield a 3 dimensional geometry - A whole new, previously unexplored mathematical space, in other words? De Morgan had already grappled with this question, and he was defeated by it. It was the Irish mathematician Sir William Rowan Hamilton who came up with the answer a few months after Ada posed the question. He had been struggling with the issue for years, and discovered that you had to go up a further dimension to 4 to come up with a workable solution. He called his new numbers quaternions, and they were to prove extremely useful in understanding the bizarre realm revealed by modern physics. [8]

Ada Lovelace

Ada suggested that if musical tones and pitches and their key relations could be expressed in a scientific language of operation we might be able to produce musical pieces on a machine. She was uniting poetry and geometry by applying different laws, where 2 parallel lines could meet:

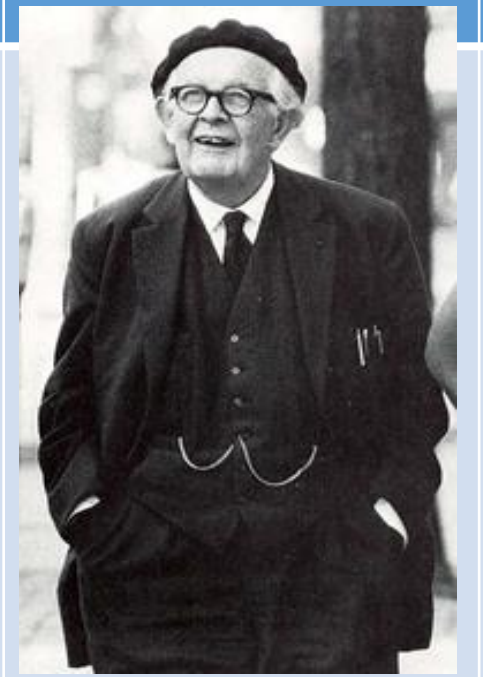
> The Analytical Engine has no pretensions whatever to *originate* any thing. It can do whatever we *know how to order it* to perform. It can follow analysis, but it has no power of *anticipating* any analytical relations or truth. Its province is to assist us I making *available* what we are already acquainted with.[9]

Rehan Qayoom, 2009

Quaternions and Maxwell (1873)

- Maxwell originally wrote his electromagnetism equations (20 of them) partly in a variation of quaternion notation, for the first two chapters, the rest in coordinate notation.
  The quaternions he used were “pure quaternions, meaning simply a vector and no use of the scalar term. He later revised his work to remove the quaternion notation entirely, since many people were unfamiliar with this notation. But he felt that quaternions were a good aid to thinking geometrically, and led to very simple expressions.
- Heaviside re-wrote the Maxwell Equations in 1893, reducing them from 20 to 4 and using vector notation. This was strongly criticized by some scientists, and was celebrated by others.
- Tesla later spent many hours reading Maxwell’s original equations, including the parts written using quaternions.

Intellectual History -- Influencers

Pioneer	Quaternion Theory of Relatives (Relations)	Models for Child Development of Logic	Octonion Advocate and Developer	Octonion Applier to Cognition and AI
				
Benjamin Peirce	Charles Sanders Peirce	Jean Piaget	John Baez	Ben Goertzel

Intellectual History -- Influencers

- **Benjamin Peirce (1809-1870)** worked with quaternions for over 30 years, starting in 1847, only 4 years after they were invented by Hamilton.
- Benjamin Peirce was the chairman of the Math Dept. and professor at Harvard, with interests in celestial mechanics, applications of plane and spherical trigonometry to navigation, number theory and algebra. In mechanics, he helped to establish the effects of the orbit of Neptune in relation to Uranus.
- He developed and expanded quaternions into the very important field of linear algebra.
- He wrote the first textbook on linear algebra during 1870-1880, thereby introducing these ideas to the European continent and stressing the importance of pure (abstract) math, a value taught to him by his colleague, Ralph Waldo Emerson, as described in Equations of God, by Crowe.
- The book was edited and published posthumously by Peirce's son, Charles Sanders Peirce in 1882. (Note: He created semiotics and pragmatism.)

Intellectual History -- Influencers

Jean Piaget (1896-1980)

- Likely the greatest psychologist of Child Development of the 20th Century
- Was influenced by Charles Sanders Peirce, by revisionist mathematics (Bourbaki group), and by the philosophy of Structuralism. He was a Constructivist
- Quaternions were very useful to parts of his work, in development of logic and in development of new schemata via imbedding rather than substitution
- Wrote a philosophical novel when he was 22 (1915) about the ideas of Henri Bergson
- With Barbel Inhelder, wrote the book The Child's Conception of Space (1956), drawing on abstract math including the child's sequentially emerging understanding of the operation of topology, affine geometry, projective geometry, and Euclidean geometry

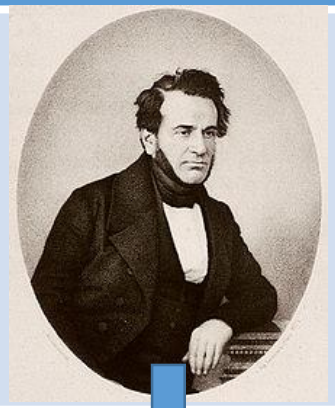
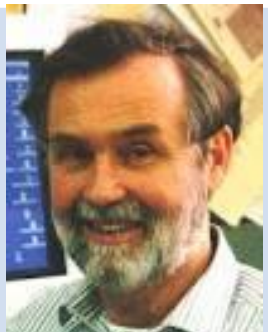
The Engines of Thought: Jean Piaget and the Usefulness of Quaternions

Piaget on the Relationship between Mind, Mathematics, and Physics

Evans: Why do you think that mathematics is so important in the study of the development of knowledge?
Piaget: Because, along with its formal logic, mathematics is the only entirely deductive discipline. Everything in it stems from the subject's activity. It is man-made. What is interesting about physics is the relationship between the subject's activity and reality. What is interesting about mathematics is that it is the totality of what is possible. And of course the totality of what is possible is the subject's own creation. That is, unless one is a Platonist.

From a 1973 interview with Richard Evans (Jean Piaget: The Man and His Ideas)

Quaternion Generalization: Clifford Algebra & Octonion Evolution

William Hamilton Quaternions, 1843	Hermann Grassmann Geometric Algebra (GA), 1840-1844	Olinde Rodrigues Theory of Rotations, (Derived from Euler's 4 squares formula), 1840	John T. Graves Octonions, 1843
			No picture available
William Clifford **Clifford Algebra, unified GA, 1878**	**David Hestenes** **Revived/restructured GA, 1950s**	**Simon L. Altmann** **Quaternions & Rotations, 1986**	**John Baez** **Octonion applications, 2002**
			

Controversies

- 1843 – 1850s (Described in book, *Equations from God,* by Daniel Cohen)
  - Quaternions are pure math; are they worth the same effort that could be given to applied math? (*Emerson urges Benjamin Peirce to say yes.)*
- 1843 – 1870s
  - Are quaternions real or nonexistent as math entities, because they occupy a 4-D home? (comparable to the algebraic space of all transformation rotations of all 3D vectors.) Is this 4D tool a mathematical reality in a 3-D world?
- 1891-1894
  - The running Grand Debate between proponents of quaternions, vectors, and coordinates
- 1880-1905
  - Should Maxwell's Equations have been re-written and simplified by Oliver Heaviside, eliminating the quaternion formulation? (*Whittaker, Tesla, L.L. Whyte, others, say no.*)

Controversies – Quaternion Advocates versus Vector Advocates

Historian Michael Crowe concludes that the development of quaternions led directly to the development of vector analysis because quaternions contained the essential ingredients for vector representation and because quaternions became known and operationally familiar, for example, to Maxwell and to Gibbs, partly through Tait, who was a classmate of Maxwell's.

Tait was more interested in mathematical physics problems and applications than was Hamilton, who died in 1865. In 1867 Tait wrote *The Elements of Quaternions*.

Vector analysis had the opportunity to develop from Grassmann's work, but that work remained mostly obscure for over 30-40 years. But it did influence Gibbs at some point, contributing some ideas to vector formulation.

Applications

BENEFITS -- Examples of Quaternion Application to Problems and Processes

ARCHETYPE – Spatial Rotation, Orientation, and Alteration of the Frame of Reference

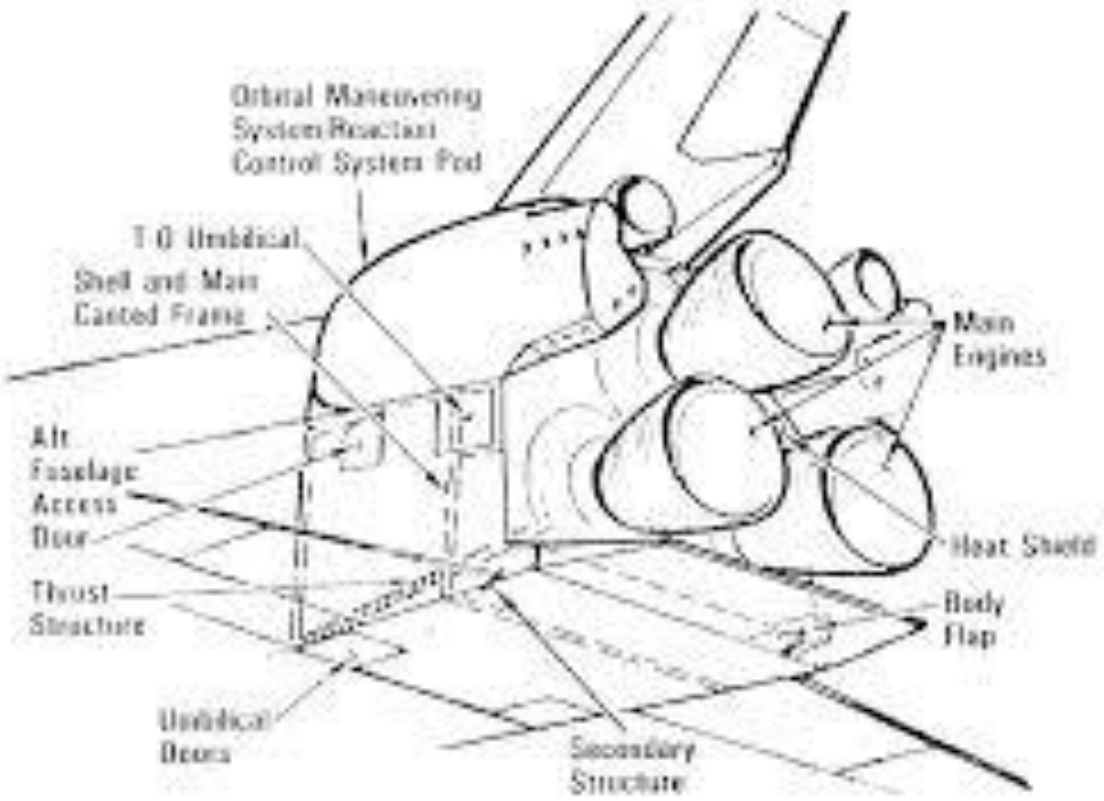
Applications – Partial List

- The list below represents a great variety of tasks and interests. Yet, their underlying functional themes are mostly **orientation, filtering, smoothing, and control**:

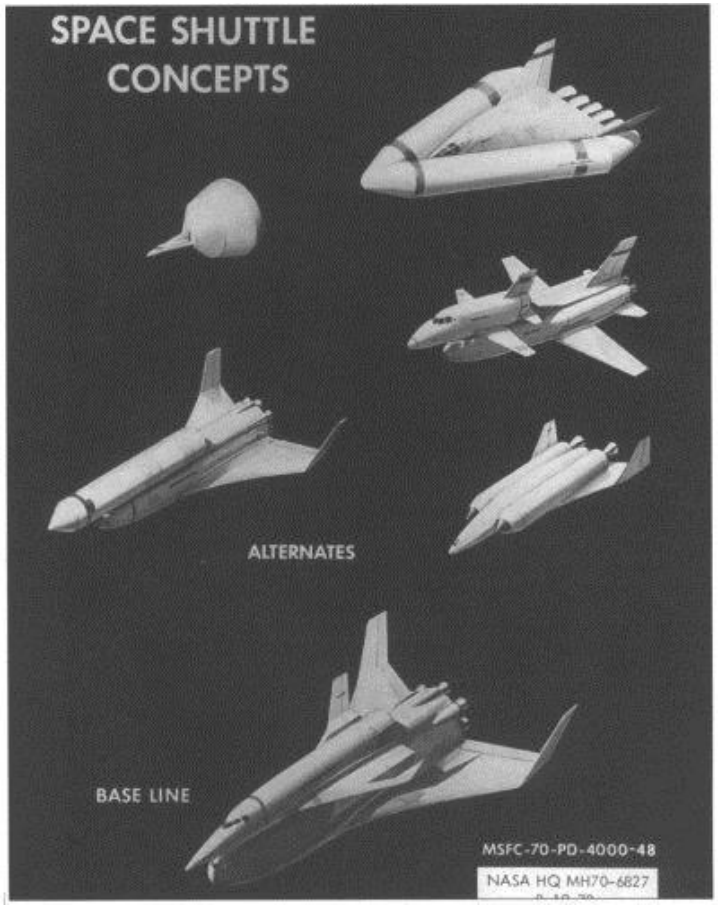
  - Virtual Reality
  - Real and mental rotation
  - Mathematical Physics problems (e.g. Maxwell Equations, quantum physics)
  - Aerospace – space shuttle pilot software
  - Computer graphics, video games, smooth interpolation
  - DNA genomic analysis
  - Bio-logging (animal locomotion orientation)
  - Music composition
  - Intellectual development of logic
  - Imbedded schema augmentation in human development
  - Eye tracking
  - Supergravity
  - Signal processing and filtering
  - Control Processing and Frame (of Reference) Control
  - Color Face Recognition
  - Quantum Physics (e.g. Dirac and Special Relativity – 2x2 Pauli Spin Matrices)

Applications - Aerospace

Applications - Aerospace



Aft Fuselage Structure



Applications – Aerospace – Elements of Movement

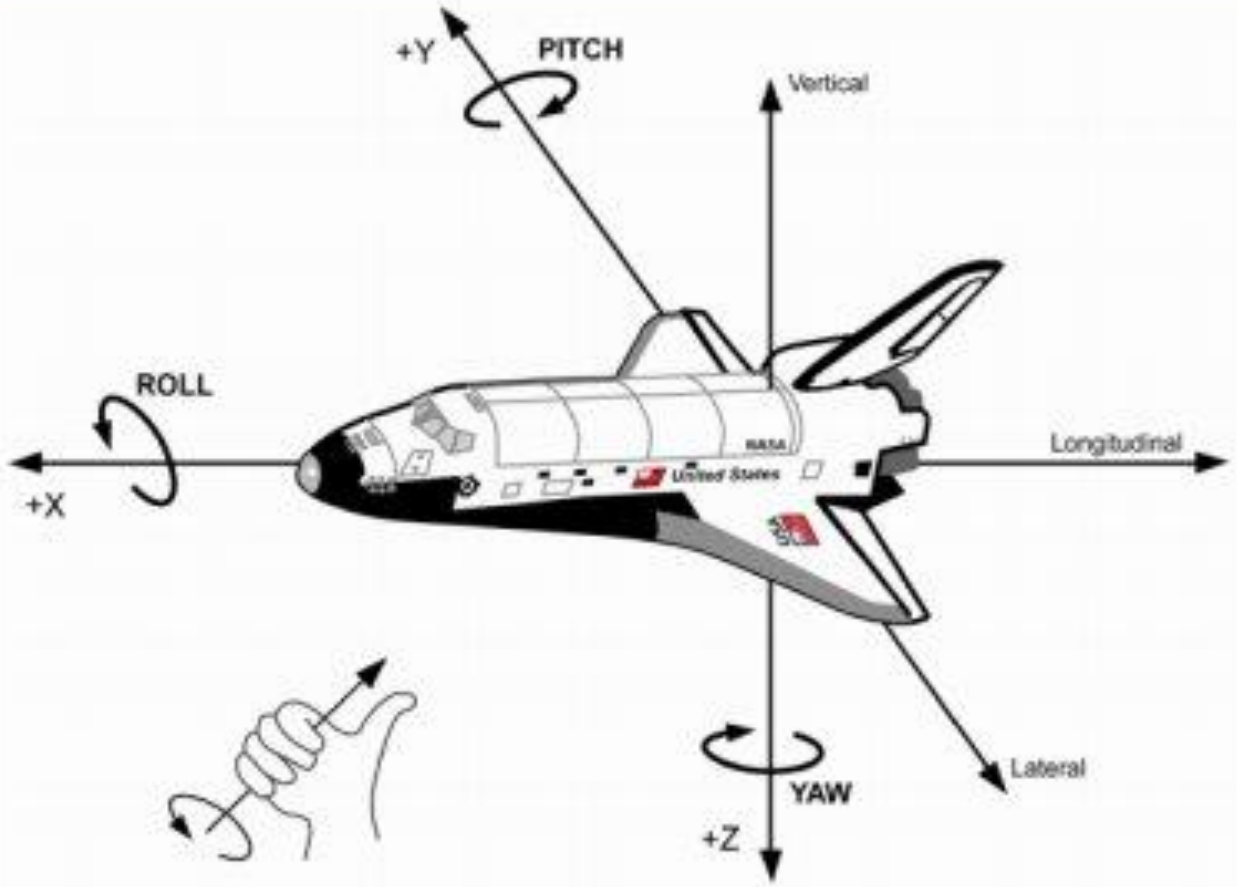


Figure 1: Diagram of the X, Y, and Z body Axes of the space shuttle. Rotation about the X-axis is called Roll, about the Y-axis is called Pitch, and about the Z-axis is called Yaw. The direction of the rotation follows the right hand rule, which states that the thumb of the right hand would be aligned with the positive axis and the direction of the rotation is positive in the direction of the fingers when curling around the axis. The arrows show positive rotation.

Applications – Aerospace Guidance

- Guidance equipment (gyroscopes and accelerometers) and software first compute the location of the vehicle and the orientation required to satisfy mission requirements.
- Navigation software then tracks the vehicle's **actual location and orientation**, allowing the flight controllers to use hardware to transport the space shuttle to the **required location and orientation.** Once the space shuttle is in orbit, the Reaction Control System (RCS) is used for **attitude control**.
- Attitude is the orientation the space shuttle has relative to a frame of reference. The RCS jets control the attitude of the shuttle by affecting rotation around all three axes.
- Three terms, pitch, yaw, and roll, are used to describe the space shuttle's attitude. Moving the nose up and down is referred to as "pitch," moving the nose left and right is referred to as "yaw," and rotating the nose clockwise or counterclockwise is referred to as "roll" (Figure 1)."

- From: http://www.nasa.gov/pdf/519348main_AP_ST_Phys_RollManeuver.pdf

Applications – Aerospace
Quaternion Advantages – Compact, Transparent

There are three historical ways to perform a mathematical rotation of a 3D object:

-- orthogonal matrix,

-- Euler angle

-- quaternion

- The representation of a rotation as a quaternion (4 numbers) is more compact than the representation as an orthogonal matrix (9 numbers).
- Furthermore, for a given axis and angle, one can easily construct the corresponding quaternion, and conversely, for a given quaternion one can easily read off the axis and the angle. Both of these are much harder with matrices or Euler angles.

- (Wikipedia)

Applications – Aerospace
Quaternion Advantages – Reduce Errors

- When composing several rotations on a computer, rounding errors necessarily accumulate. A quaternion that's slightly off still represents a rotation after being normalised: a matrix that's slightly off may not be orthogonal anymore and is harder to convert back to a proper orthogonal matrix.
- Quaternions also avoid a phenomenon called gimbal lock which can result when, for example in pitch/yaw/roll rotational systems, the pitch is rotated 90° up or down, so that yaw and roll then correspond to the same motion, and a degree of freedom of rotation is lost. In a gimbal-based aerospace inertial navigation system, for instance, this could have disastrous results if the aircraft is in a steep dive or ascent. *This danger was portrayed in the film, Apollo 13.*
- (Wikipedia)

Applications – Celestial Mechanics

USING QUATERNIONS TO **REGULARIZE** CELESTIAL MECHANICS
(avoiding paths that lead to collisions)

“Quaternions have been found to be the ideal tool for developing and determining the theory of spatial regularization in Celestial Mechanics.”

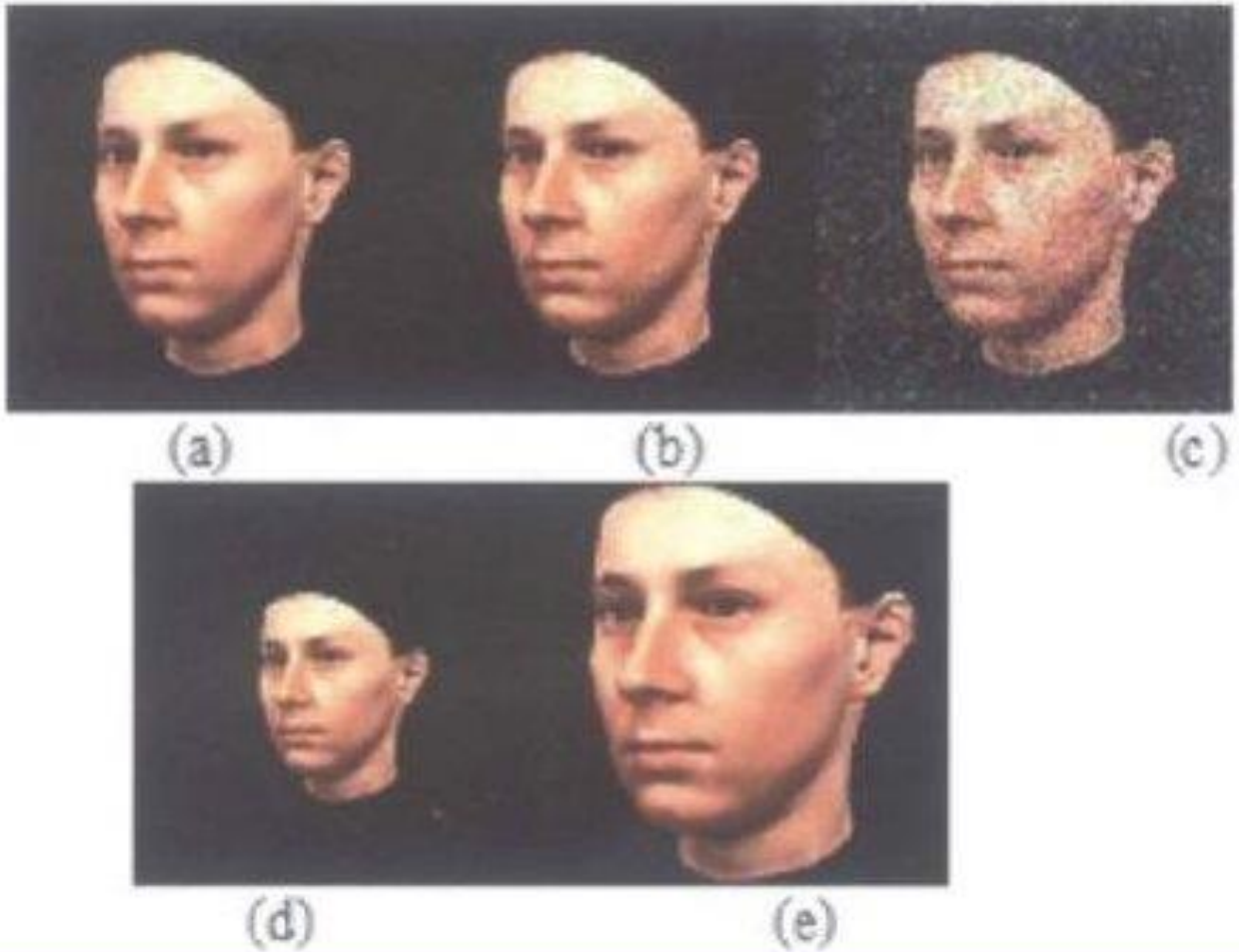
Waldvogel, Jorg (2008). Quaternions for regularizing Celestial Mechanics: The right way. Celestial Mechanics and Dynamical Astronomy, 102: 149-162

Applications – Computer Graphics

- In video games and other applications, one is often interested in "smooth rotations", meaning that the scene should slowly rotate [instead of jumping] in a single step.
- This can be accomplished by choosing a curve such as the spherical linear interpolation in the quaternions, with one endpoint [of the curve] being the identity transformation 1 (or some other initial rotation) and the other being the intended final rotation.
- **This is more problematic with other representations of rotations.** (Wikipedia)

Applications – Color Face Recognition / Pattern Recognition Quaternion Advantages: Speed, Accuracy (Wai Kit Wong)

Figure 10.12 An example of a person set (a) original image (b) embedded with mild salt and pepper noise, (c) embedded with heavy salt and pepper noise, (d) shrink (e) dilation.



Applications – Color Face Recognition / Pattern Recognition Quaternion Advantages: Speed, Accuracy (Wai Kit Wong)

Table 10.2 Normalized enrollment stage time consumption, normalized classification stage time consumption, and matching accuracy for different color face classification method

Color face classification method	Enrollment stage normalized time consumption (for training all data sets in database)	Classification stage normalized time consumption (for matching 10,000 tested image)	Accuracy (output human names/ID match with the correspondence input images)
Conventional NMF	2.76	1.39	80.18%
BDNMF	3.51	1.55	83.37%
Hypercomplex Gabor filter (Mahalanobis distance classification)	1.36	1.20	86.13%
Quaternion-based fuzzy neural network classifier	1.00	1.00	92.06%

Applications – Color Representation and Image-Signal Processing

PREVENTING HUE DISTORTION

In classical image filtering, we typically have to handle the problem of color space *closure*. Image pixel values have a finite range, typically positive and scaled to the interval [0,1]. If a filtering operation results in a pixel value outside this range, then it must be brought back into the range, ideally without distortion artifacts. One says the range of legitimate pixel values is not closed with respect to the filtering operation. For grayscale images, closure is typically forced by clipping the pixel value at the range boundaries. For color image filters, the color space is a 3D bounded volume, e.g., the RGB color space is typically mapped to a unit edge-length cube with one corner (the black color) at the origin. The color space closure problem becomes more difficult, especially if we wish to avoid visual distortions in the color image output. To reduce these distortions, the origin is moved to the center of the color cube. This means that all pixel values are now vectors pointing away from mid-gray, the center of the cube. Now closure can be enforced by clipping output pixel values to the value where the pixel vector passes through the cube's surface. This three-space clipping eliminates hue distortions by holding the orientation of the pixel vector constant during the clipping operation, i.e. only the length of the vector is altered [SAN 04].

Ell, T., Le Bihan, N., and S. Sangwine (2014). Quaternion Fourier Transforms for Signal and Image Processing. Wiley.

Applications –Archetype Relationship of Signal Processing to Orientation Change in Hypercomplex Analysis

(1) *Archetype* quaternion and hypercomplex processes are used to change the orientation of an object or a viewing frame. How does this relate to quaternion signal processing abilities?

(2) Hypercomplex approaches [to signal processing], including using quaternions, succeed because **they can effectively control the frame of reference to best identify the information in the signal.** This is yet another application of their ability to relate to orientation questions.

- *Book Reference:*
  Dutkay, D.E. and P.E.T. Jorgensen (2000) in Daniel Alpay (ed) (2006). Wavelets, Multiscale Systems, and Hypercomplex Analysis, page 88.
- *Online reference:*
  books.google.com/books?isbn=3764375884

Applications – Bio-logging
Energy Expenditure of Animals

BIO-LOGGING, SENSORS, AND QUATERNION-BASED ANALYSIS – Dynamic Body Acceleration

- **ABSTRACT** This paper addresses the problem of rigid body orientation and dynamic body acceleration (DBA) estimation. **This work is applied in bio-logging, an interdisciplinary research area at the intersection of animal behavior and bioengineering.**

  The proposed approach combines a quaternion-based nonlinear filter with the Levenberg Marquardt Algorithm (LMA). The algorithm has a complementary structure design that exploits measurements from **a three-axis accelerometer, a three-axis magnetometer, and a three-axis gyroscope.** Attitude information is necessary to calculate the animal's DBA [dynamic body acceleration] **in order to evaluate its energy expenditure**.

- Journal Reference:
- Hassen Fourati, Noureddine Manamanni, Lissan Afilal, Yves Handrich (2011). **A Nonlinear Filtering Approach for the Attitude and Dynamic Body Acceleration Estimation Based on Inertial and Magnetic Sensors: Bio-Logging Application. IEEE Sensors Journal, 11,1: 233-244**

Applications – Bio-logging
Motion Capturing and Analysis

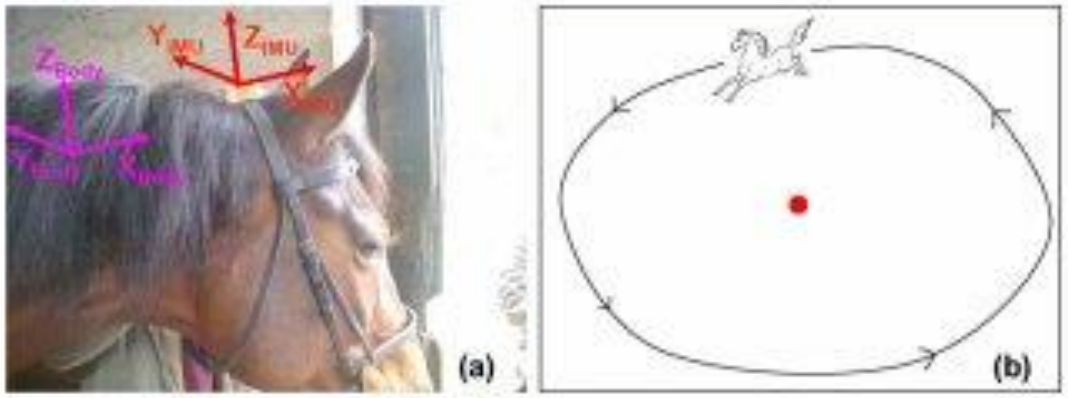


Fig. 12. (a) MTi-G attached to the head of the horse. (b) Schematic diagram of how the horse performed its motion.

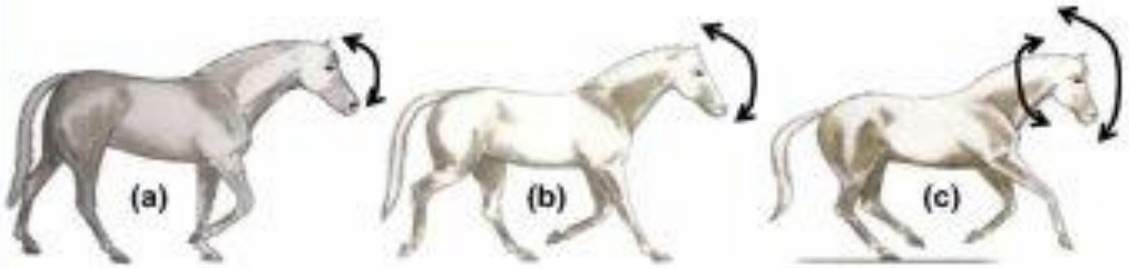


Fig. 13. Gaits of the horse and the movement performed by the head. (a) Walk. (b) Trot. (c) Gallop.

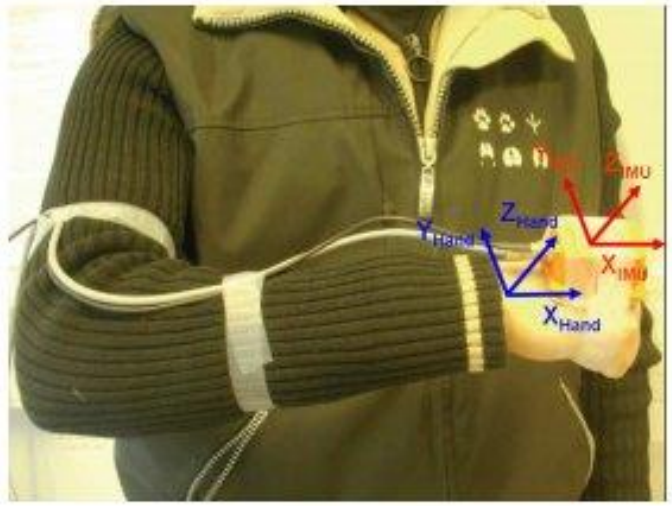


Fig. 9. Subject with the MTi-G attached to the hand.

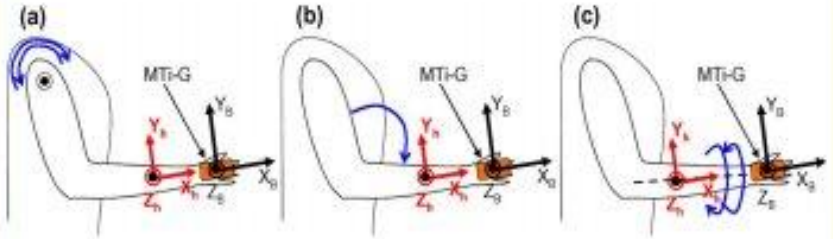


Fig. 10. Exercises performed during the hand motion. (a) Clockwise and anti-clockwise shoulder rotation. (b) Elbow extension. (c) Clockwise and anti-clockwise rotation around the dashed line axis defined along the forearm segment.

Applications – Bio-logging
3D Analysis Gives Better Results Than 2D, and Quaternions Excel in 3D Motion Analysis

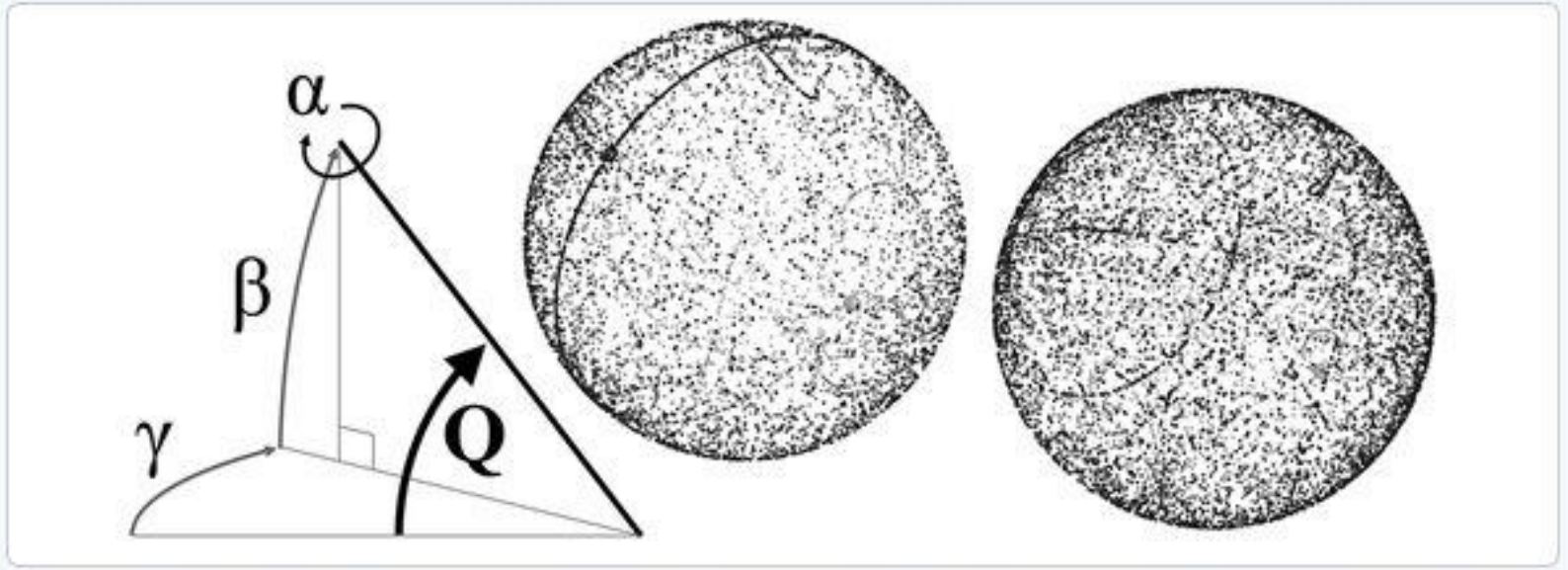
BODY ATTITUDE AND DYNAMIC BODY ACCELERATION IN SEA ANIMALS

- "**Marine animals** are particularly hard to study during their long foraging trips at sea. However, the need to return to the breeding colony gives us the opportunity to measure these different parameters using bio-logging devices."
- "Note that the use of inertial and magnetic sensors is relatively recent, due to the difficulty to develop **miniaturized technologies** due to **high rate record sampling** (over 10-50 Hz)."
- "**The obvious advantage to this new approach is that we gain access to the third dimension space**, which is a key to a good understanding of the diving strategies observed in these predators…"

Hassen Fourati et al, A quaternion-based Complementary Sliding Mode Observer for attitude estimation: Application in free-ranging animal motions.

Applications – Pharmaceutical Molecules and Receptor Docking

Abstract



Much work has been done on algorithms for structure-based drug modeling in silico, and almost all these systems have a core need for three-dimensional geometric models. The manipulation of these models, particularly their transformation from one position to another, is a substantial computational task with design questions of its own. Solid body rotation is an important part of these transformations, and we present here a careful comparison of two established techniques: Euler angles and quaternions. The relative superiority of the quaternion method when applied to molecular docking is demonstrated by practical experiment, as is the crucial importance of proper adjustment calculations in search methods.

Applications – Pharmaceutical Molecules and Receptor Docking

- QUATERNION ANALYSIS OF MOLECULE MANEUVERING AND DOCKING
- Article: "Doing a Good Turn: The Use of Quaternions for Rotation in Molecular Docking"
- it parallels quaternion uses in studying animal motion and space shuttle flight
- http://pubs.acs.org/doi/abs/10.1021/ci4005139 Oxford research team

- **Skone, Gwyn, Stephen Cameron *, and Irina Voiculescu (2013)** Doing a Good Turn: The Use of Quaternions for Rotation in Molecular Docking. J. Chemical Information and Modelling (ACS), 53(12), 3367-3372

Applications – Organic Chemistry

Tetrahedron structure and quaternion relationships

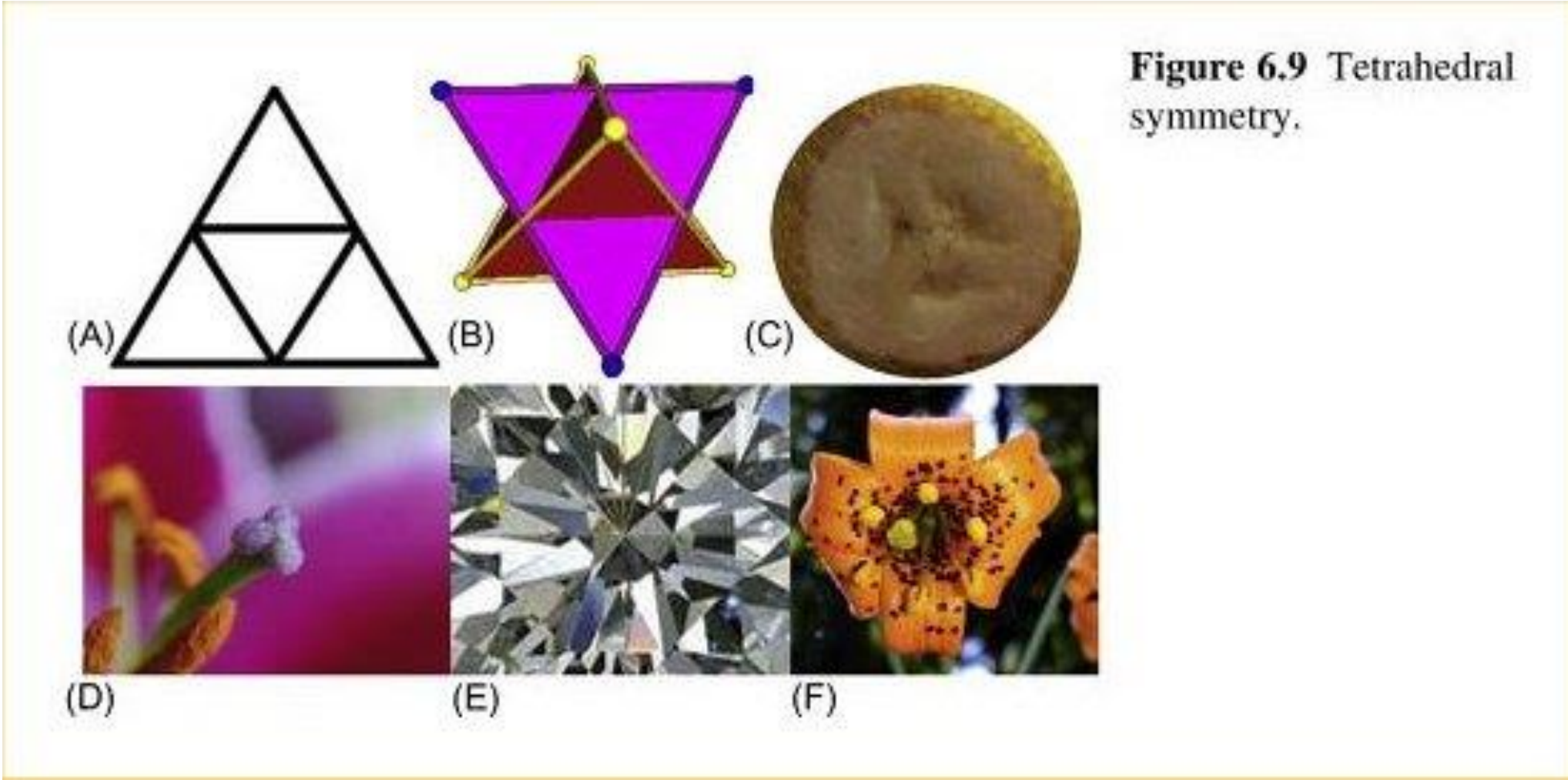


Figure 6.9 Tetrahedral symmetry.

Applications – Organic Chemistry
Methane, Ammonia, and Tetrahedron Structure

Tetrahedron structure and quaternion relationships

- "A leading journal in organic chemistry is called "Tetrahedron" in recognition of the tetrahedral nature of molecular geometry."
- "Found in the covalent bonds of molecules, tetrahedral symmetry forms the methane molecule (CH_4) and the ammonium ion (NH_4^+) where four hydrogen atoms surround a central carbon or nitrogen atom."
- "Italian researchers Capiezzolla and Lattanzi (2006) have put forward **a theory of how chiral tetrahedral molecules can be unitary quaternions**, dealt with under the standard of quaternionic algebra."

Dennis, L., et al (2013), The Mereon Project: Unity, Perspective, and Paradox.
Capozziello, S. and Lattanzi, A. (2006). Geometrical and algebraic approach to central molecular chirality: A chirality index and an Aufbau description of tetrahedral molecules.

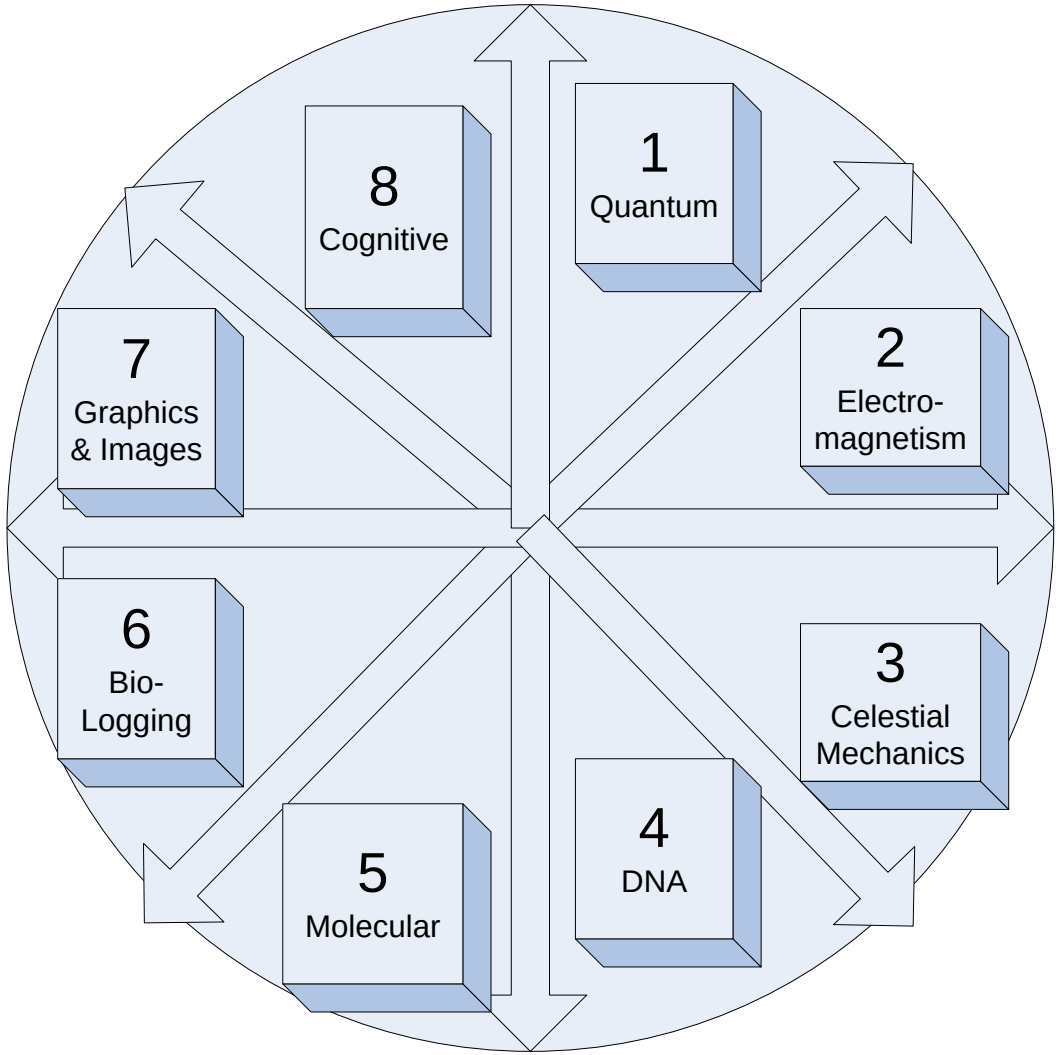
Applications - Quantum Mechanics

- Objects related to quaternions arise from the solution of the Dirac equation for the electron. The non-commutativity is essential there.
- The quaternions are closely related to the various “spin matrices” or “spinors” of quantum mechanics.

References:

- White, S. (2014). Applications of quaternions. www.zipcon.net
- Finkelstein, Jauch, Schiminovich, and Speiser *Foundations of Quaternion Quantum Mechanics*, J. Math. Phys, **3** (1962) 207-220

Applications – Represent All Levels of Nature



Math

Quaternion Neighborhood in Math

INRC Math “Group” (tessarines)

Defining Properties of Quaternions (3 imaginaries)

Rotation and Angles

Generalization to Octonions (7 imaginaries), and Fano Plane as a Bridge between Quaternion Algebra and Projective Geometry

Math Neighborhood

Branches of Math --

Analysis

(calculus, limit processes)

Algebra

(combining elements, performing symbol operations, solving equations)

Geometry

(Roles and Relationships .. e.g. Lines and points, reflection and rotation, trajectory, spatial, inside, reversal, intersection)

Math Neighborhood

Examples of Number Systems –

Natural numbers

Whole numbers

Integers

Rational numbers

Real numbers

Complex numbers

Extended Math Neighborhood

(Hierarchical – each imbedded in next)

Natural numbers

Whole numbers

Integers

Rational numbers

Real numbers

Complex numbers

Hypercomplex Numbers:

Quaternion numbers

Octonion numbers

Geometric Algebra*

Clifford Algebra systems

*A Clifford algebra of a finite-dim. vector space over the field of real numbers endowed with a quadratic form

Note: Hypercomplex number definition – its components include **multiple** kinds of imaginary numbers

Algebraic Math Neighborhood

Some Categories of Algebraic Systems –

Groups – one operation, with inverses, closure
Fields – 2 operations, each with inverses
Rings – Field with unique inverses defined for all but zero element
Algebras – ring with dot-product multiplication

A Powerful Type of Algebra**: The Normed Division Algebra.**

- There are only four of them.
- They are nested inside of each other:

-- Real (1D)
-- Complex (2D)
-- Quaternions (8 elements) (4D)
-- Octonions (16 elements) (8D)

Math Neighborhood – A Special Hypercomplex Group

INRC group
(4 elements)

Other names:

- Tessarine
- Klein 4-group

Piaget and the INRC Group:
Operations at the Foundation of Traditional Logic

Jean Piaget (1896-1980) [*from webpage of Alessio Moretti*, http://alessiomoretti.perso.sfr.fr/NOTPiaget.html]

The Swiss psychologist Jean Piaget, one of the leading figures of "structuralism",
on top of his studies on the evolutionary construction of child cognition has proposed a model of the "logical capacities".
This is a set of 4 mental operations [on propositions], mutually related by composition laws constituting a mathematical structure of a group, namely a particular decoration of the "Klein 4 group", called by Piaget, because of the 4 operations constituting it, an "INRC group".

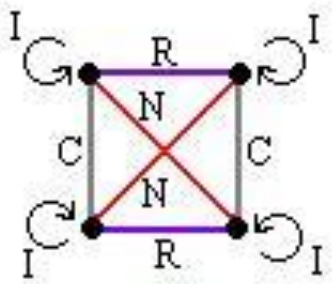
I = identity
N = inversion (negation)
R = reciprocation
C = correlation

basic mental logical operations of young humans

N = RC
R = NC
C = NR
I = NRC

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

Piaget's composition laws making the INRC structure a group



the strange but elegant square structure of the INRC group

Definition of the Unit Quaternion Group

- Cousin to the quaternion group – the INRC group (Klein 4 group).
  - Elements: 1, i, j, k (identity and three axes)
  - Rules of Combining:
    - **$i^2=j^2=k^2 = 1$,**
    - i times j=k, (NxR=C) -- negating and reciprocating proposition K
    - Triangle arrangement of elements …… I J
    - Kids develop understanding of the relationships between logical operations

- Quaternion Group: The above element plus their negatives
    - **$i^2=j^2=k^2 = -1$, -- three different square roots of minus one!**
    - i times j=k,
    - i times j = --j times I
    - 4-D Space of Rotations of 3-D Objects (and 4D objects, too!)

Definition of the Quaternion Algebra Space (By Application of Linear Algebra)

- Let us create full quaternion spaces, not just unit-length axis groups.
- These are formed out of linear combinations of the quaternion group elements 1, i, j, k, using real-number coefficients:

$$A + Bi + Cj + Dk$$

EXAMPLES:

- 3 i + 10 j -2 k + 17 is a quaternion space element.
  Note: It represents an actual specific rotation.

- In this space, the elements 1, i, j, k are called basis elements (or simply a “basis”) that generate the space through linear combinations.

Quaternions and 4D Spaces: Interpreting the Parameters of Object Rotation

- Any real-number 4D space can be interpreted as a quaternion algebra space.
- Any set of quadruple coordinates (w, x, y, z) represents a point in 4D space and can be interpreted as a 4D rotation and size-expansion transformation.
- Any set of quadruple coordinates that represents a unit (length) quaternion (therefore an element of a unit hypersphere of radius 1) can be interpreted as a rotation of a 3D subset of the 4D space:

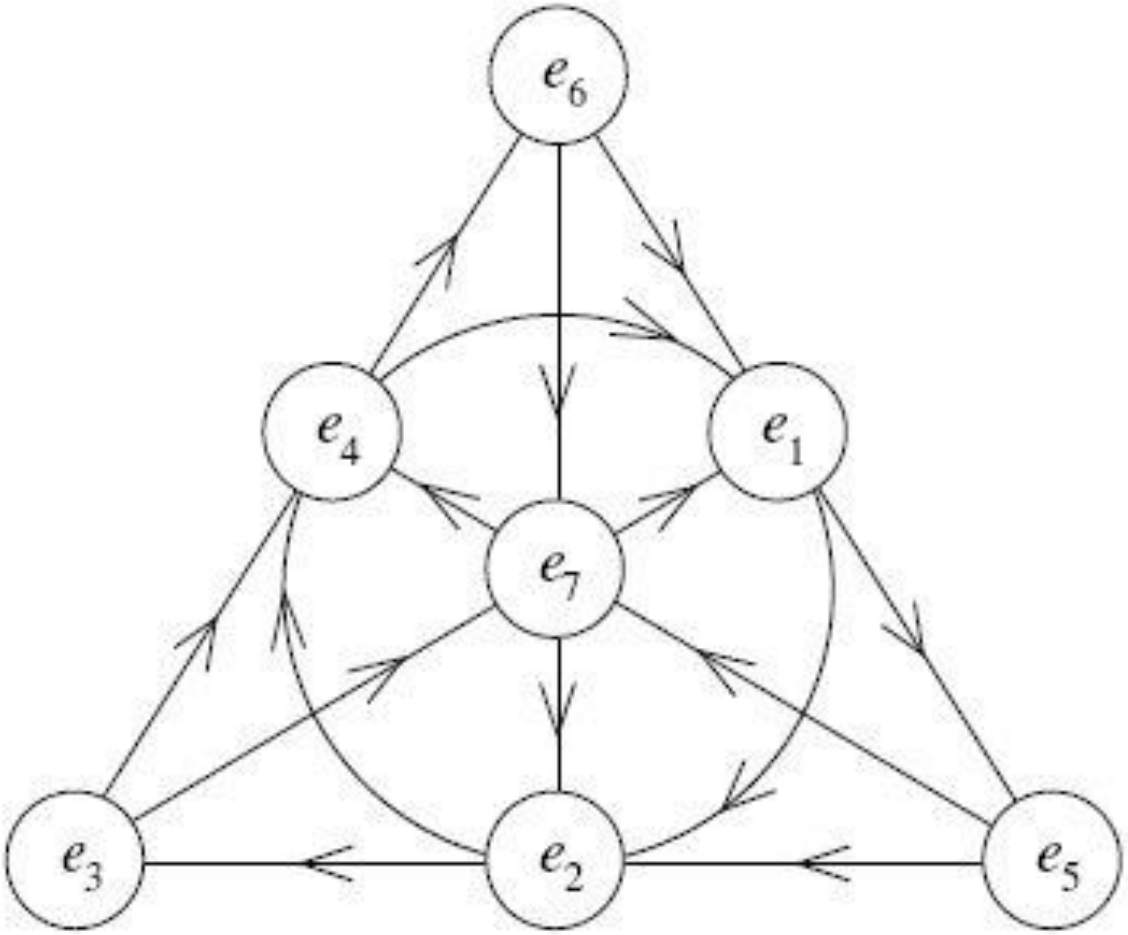
  - If (w, x, y, z) is the quadruple coordinate and point (x, y, z) has "taxi-metric" distance of 1 from the origin in a 3D subspace,
  - Then w is the angle of rotation expressed in radians,
  - Matrix representations of rotations do not have this transparency advantage of directly showing the angle being rotated.

Octonions

- Invented by William T. Graves in 1843.
- Popularized and developed further by **John Baez** during the last 15 years (ref. online videos).
- Octonion Elements: seven independent axes and identity element (1) in an 8-dimensional space.
  - 1, e1, e2, e3, e4. e5, e6, e7 and their negatives.
  - Multiplication is not associative.
- These elements, without the 1 element and the negative elements, form the smallest example of a projective geometry space, the 7-element Fano plane.
- The Fano plane is a GRAND BRIDGE between quaternion algebra and projective geometry!

Fano Plane -- Coding

- Fano Plane coding is a very efficient way of coding items for computer storage



Music Cognition and 4D

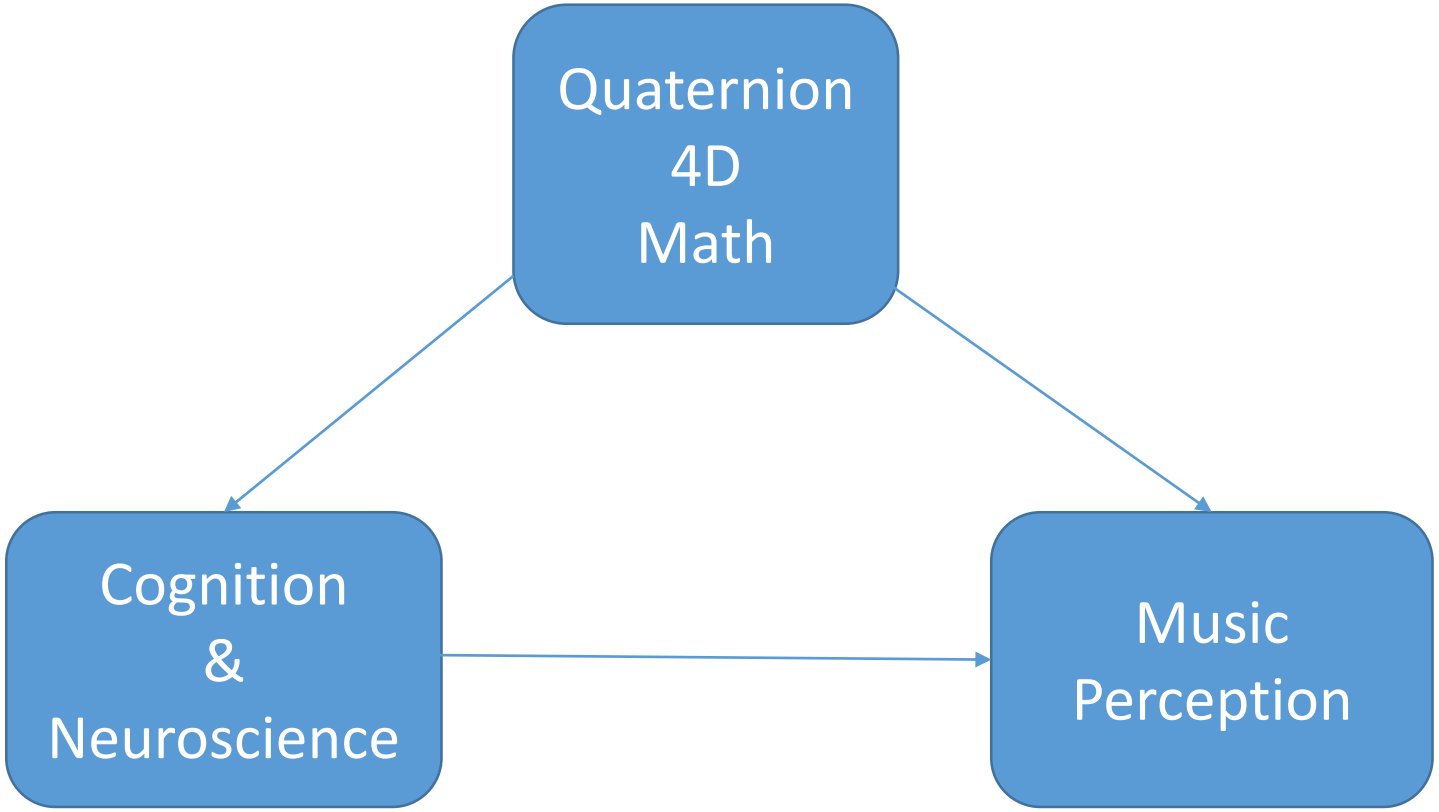
Perspective from Social Anthropology –
Internal and External Tools – Perception, Expression, Art, Music, Thinking –
Melody of the Text

Perspective from Modern Neuroscience

Music Theory, Psychology, and 4D Math

Music as a Virtual-Reality Tool

The Fertile Triangle



Introduction

How do the pieces of spatial and music cognition research fit together?

SOME THEORIZING:

What are the cognitive dimensionalities of musical objects such as melodies, notes, and chords – particularly melodies? 1D, 3D, 4D, 8D? Other? For melodies, my hunch is 4D and/or 8D.

A melody is a whole, not just a sequence.

Ben Goertzel believes that short-term memory is octonion (8D) in character, because of its capacity for creating temporal reversal, and because it typically can hold 7 independent elements at one time.

If melodies are analogous to the simultaneous contents of short-term memory, then they are likely to be 8D. This is consistent with Arnold Trehub saying that sentences and tunes are meaningful because they benefit from the effects of short-term memory of the autaptic cells, if we focus on successive buildup of parts (4D) of the melody to make up the whole (8D) for a prefrontal cortex general short-term memory environment.

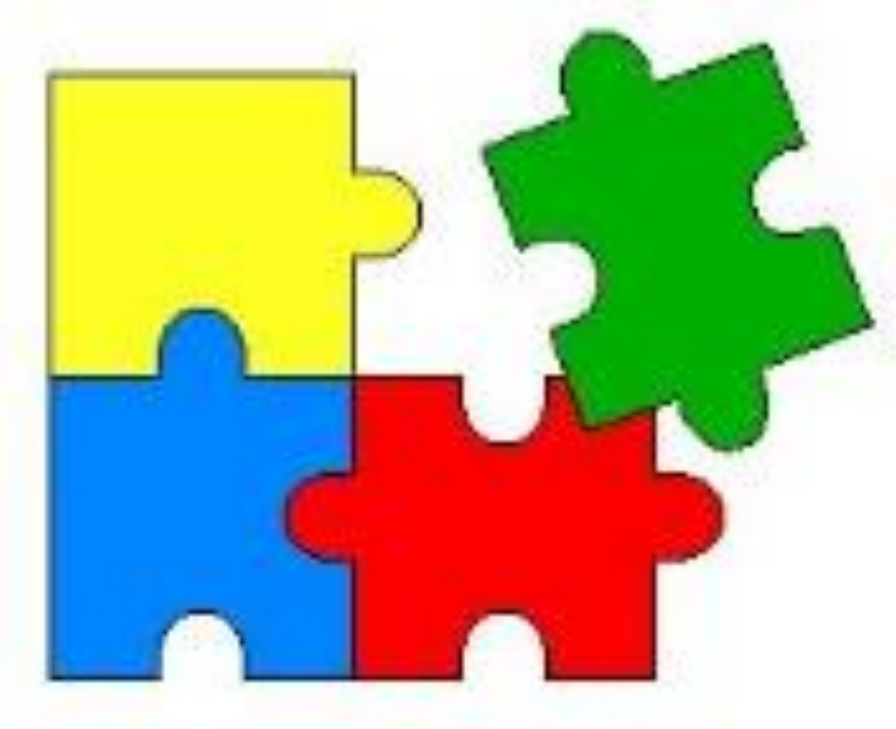
Let’s look at some relevant research.

3D **General** Cognition Models

3D Virtual Retinoid Space with Self in Center
(Arnold Trehub, 1991, 2005)

Default 3D Multisensory Space in Parietal Lobe, supported by thalamus
(Jerath and Crawford, 2014)

Supramodal Mental Rotation of Melody and Visual Objects in Parietal Lobe
(Marina Korsakova-Kreyn, 2005)



4D **Music** Cognition Models

4D Distances of Musical Keys From Each Other (Krumhansl & Kessler, 1982)
Possible 4D Nature of Melodies? (Gilles Baroin, 2011; others)
4D/5D Melody of the Text (Mike Mair, 1980)

General Cognition and Music Cognition

GENERAL COGNITION **-- SPATIAL AWARENESS**, PERCEPTION, PROCESSING

- Parietal lobe has two parts: inferior (IPL) and superior (SPL).
  My interpretation:
  - IPL – spatial display
  - SPL – perception of motion, coordination of action

- Multisensory, supramodal processing in parietal lobe, and perception of real and imagined (virtual) objects and perspectives -- 3D (4D)
  - Trehub (2005) - IPL (consciousness)
  - Jerath & Crawford (2014) IPL (connection to consciousness via thalamus)
  - Korsakova-Kreyn (2005) – SPL (mental rotation)
  - Daniel Wolpert (2014) – SPL (sensorimotor integration; “why do we have brains – to control motion”)
- Self at center of surrounding space (consciousness – Damasio, Trehub)

General Cognition and Music Cognition

MUSIC COGNITION – **HARMONY** SYSTEMS
-- OUR FOCUS BECAUSE OF ITS CENTRALITY TO MELODY AND MUSIC

- Notes – tonal attraction – gravity model (gives potential values to each tone for movement toward the tonic note)
- Music in the brain versus in the air:
  - Acoustics – Sound in the Air
  - Acousmatics – Sound in the Brain – **This one is our interest**.
    Note: Dimensionalities of objects may be different than in acoustics.

General Cognition – Trehub Retinoid Model

Here are Arnold Trehub's views on the potential of the retinoid space in the brain to provide 4D capabilities:

> "I'm not knowledgeable enough to respond to your detailed observations about music, but I must point out that all autaptic-cell activity in retinoid space is 4D because autaptic neurons have short-term memory.
>
> This means that there is always some degree of temporal binding of events that are "now" happening and events that happened before "now". The temporal span of such binding probably varies as a function of diffuse activation/arousal.
>
> The temporal envelope of autaptic-cell excitation and decay defines our extended present. This enables us to understand sentences and tunes."
>
> *Via email*

General Cognition – Trehub Retinoid Model

- Two key assumptions of the retinoid model are:

  (1) visually induced neuronal excitation patterns can be spatially translated over arrays of spatiotopically organized neurons, and
  (2) excitation patterns can be **held in short-term memory within the retinoids** by means of self-synapsing neurons called autaptic cells.
- **I made these assumptions originally because they provided the theoretical grounding for a brain mechanism capable of processing visual images in 3D space very efficiently and because they seemed physiologically plausible (Trehub, 1977, 1978, 1991).**
- **More recent experimental results provide direct neurophysiological evidence supporting these assumptions.**

  Arnold Trehub: Space, Self, and the Theater of Consciousness (2005)

General Cognition – Trehub Retinoid Model

General observations:

- This hypothesized brain system has structural and dynamic properties enabling it to register and appropriately integrate disparate foveal stimuli into a perspectival, egocentric representation of an extended 3D world scene including a neuronally tokened locus of the self which, in this theory, is the neuronal origin of retinoid space.
- As an integral part of the larger neuro-cognitive model, **the retinoid system is able to perform many other useful perceptual and higher cognitive functions.** In this paper, I draw on the hypothesized properties of this system to argue that **neuronal activity within the retinoid structure constitutes the phenomenal content of consciousness and the unique sense of self** that each of us experiences.

ResearchGate.net
Where I Met Arnold Trehub and Many Others

- Free, minimal requirements
- Paper repository
- Lively question discussion groups
- 5 million members
- Heavily international
- Internal messaging is available between members

General Cognition – Jerath & Crawford Parietal/Thalamus Model

Jerath, R. and Crawford, M. W. (2014). Neural correlates of visuospatial consciousness in 3D default space: Insights from contralateral neglect syndrome. Consciousness and Cognition, 28, 81–93.

Summary:

- We propose that the thalamus is a central hub for consciousness.
- We use insights from contralateral neglect to explore this model of consciousness.
- The thalamus may reimage visual and non-visual information in a 3D default space.
- 3D default space consists of visual and other sensory information and body schema.

General Cognition – Jerath & Crawford Parietal/Thalamus Model

One of the most compelling questions still unanswered in neuroscience is how consciousness arises.

In this article, we examine visual processing, the parietal lobe, and contralateral neglect syndrome as a window into consciousness and how the brain functions as the mind and we introduce a mechanism for the processing of visual information and its role in consciousness.

We propose that consciousness arises from integration of information from throughout the body and brain by the thalamus and that the thalamus reimages visual and other sensory information from throughout the cortex in a default three-dimensional space in the mind.

We further suggest that the thalamus generates a dynamic default three-dimensional space by integrating processed information from corticothalamic feedback loops, creating an infrastructure that may form the basis of our consciousness. Further experimental evidence is needed to examine and support this hypothesis, the role of the thalamus, and to further elucidate the mechanism of consciousness.

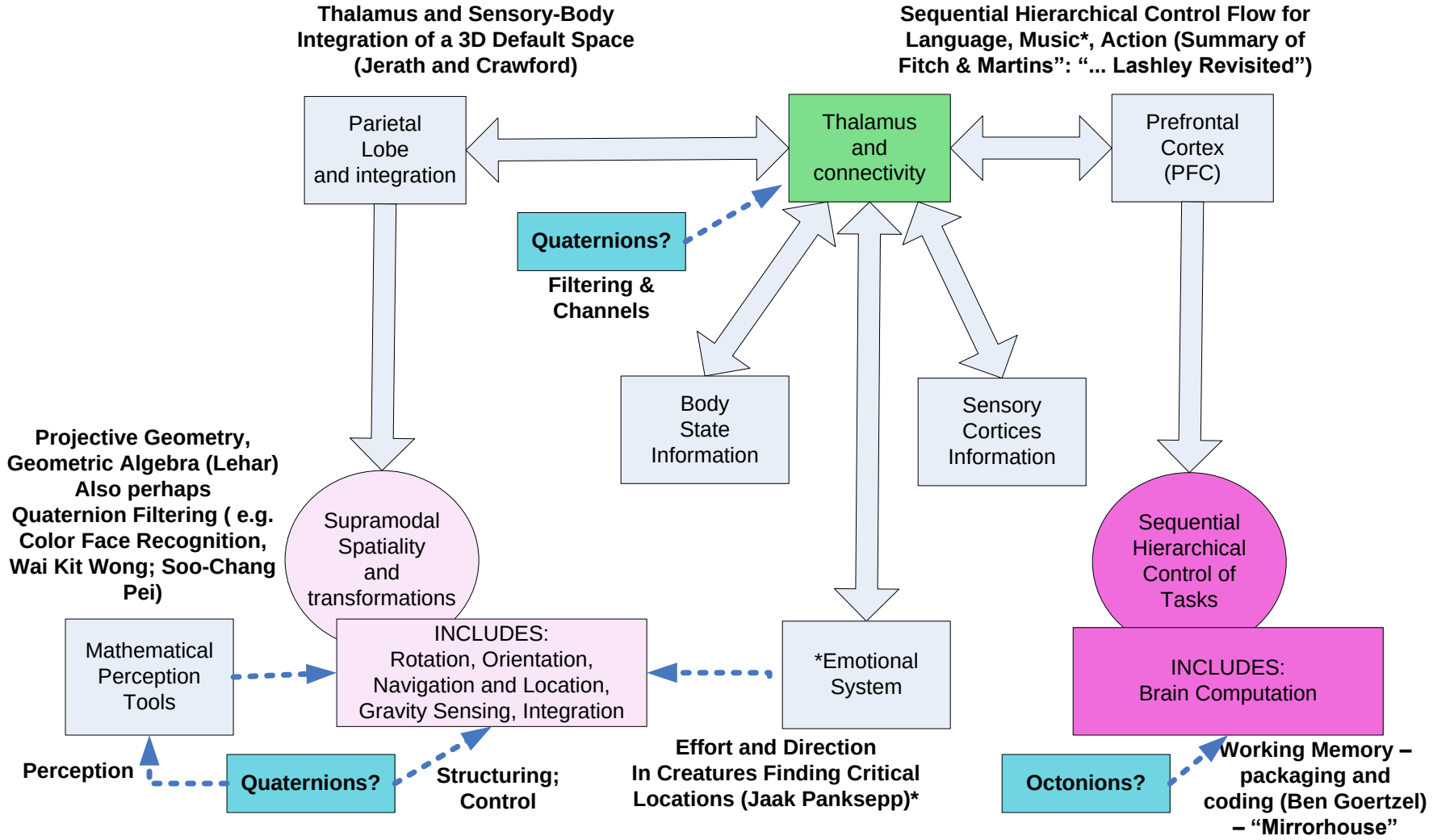
General Cognition – Korsakova-Kreyn 3D/Parietal/Supramodal Model Based on Mental Rotation

- The parietal lobes interpret sensory information and are concerned with the ability to carry out and understand spatial relationships. It was found that the right superior parietal lobe plays an essential role in mental rotation (Harris & Miniussi, 2003; Alivastos, 1992). There is neurophysiological evidence that lesions to the right parietal lobe impair mental rotation abilities (Passini et al, 2000) and that the superior parietal region seems to play a “major role in the multiple spatial representations of visual objects” Jordan et al (2001).
- **I hypothesize that perhaps the brain reads both music and spatial information as a signal-distribution within system of reference notwithstanding the modality of the signal. Recent imaging studies suggest that the parietal lobe is an integral part of a neural lateral prefrontal–parietal cortices circuit that is critical in cognition.**

The Thalamus and Its Interconnections with the Parietal Lobe, Supramodal 3D Space, and the Prefrontal Cortex

Thalamus provides Flow Path for Music
(Jaschke)

"This door is the *thalamus*, which in a musical context is initially filtering out or rather channeling certain information, before it is cerebrally processed."



Conjecture:
Possible Dimensionality Roles of Three Connected Neural Structures

1. Parietal Lobe – 3D/4D (consistent with quaternions) – spatial-multisensory display and transformation function. Activities seem to be:
   - Superior parietal lobe – motion, rotation, sensorimotor integration (*Wolpert model, Korsakova-Kreyn research*)
   - Inferior parietal lobe – display and transformation (*Trehub theory, Jerath & Crawford*)
2. Prefrontal cortex (PFC), frontal cortex – 8D (consistent with octonions) - working memory (approximately 7 degrees of freedom), hierarchical-sequential planning (*applying Ben Goertzel / Herb Klitzner conjecture and Fitch, et al review of Lashley-model-oriented research*)
3. Thalamus – 4D to 8D converter and reverse, connecting the above two structures (known fact) and re-imaging the format used by one into the format of the other. (*applying Jerath & Crawford model*)

Relevance of Carl Jung and the Jungian Community

Science and math – rich involvement

- Jung-Pauli correspondence (1932-1958)
  - KML Library in NYC (Kristine Mann Library, 28 E. 39th St.)
  - Over 1200 Pauli dreams communicated to Jung
  - Occasional role reversal in commentary on science and psychology
- Quaternions and Jung
  - Quaternity, quaternio, and quaternion – meanings and translations
  - Space-time: 3-and-1 dual interpretation, 3D space vs parallel past-present-future dimensions
- Fractals and even quaternions are natural to Jungians (Terry Marks-Tarlow is a fine example)
  - Self-similarity
  - Fractal Psych, Play, and Rapport/Communication – imperfect, deep borders
  - Quaternions – higher-level view and 4D fractal zoom
  - There are fractal values between 3D and 4D (e.g. 3.743)

Relevance of Carl Jung and the Jungian Community

Terry Marks-Tarlow on quaternions and fractals:

- "Everywhere they arise, fractals occupy the boundary zone between dynamic, open processes in nature. This quality of betweenness is illuminated by a technical understanding of fractal dimensionality. Since imaginary numbers model hidden dimensionality, in the case of fractals, this consists of infinite expanses, or imaginary frontiers that lurk in the spaces between ordinary, Euclidean dimensions. Clouds are zero-dimensional points that occupy three-dimensional space, coastlines one-dimensional lines that occupy two-dimensional planes, and mountains twodimensional surfaces draping a three-dimensional world.
- **Quaternions are products of the hypercomplex plane consisting of one real and three imaginary axes. If imaginary numbers do relate to abstract processes in consciousness, and more specifically to the fuzzy zone between body and mind, then because they are three-dimensional shadows of four-dimensional space, quaternions may provide some clues as to the internal landscape of higher dimensional thought."**

Semiotic Seams: Fractal Dynamics of Re-Entry (2004)

Quaternions and Neuroscience, Computation, and Transformation

-- Do quaternion-like mechanisms actually exist in the brain?

-- How might quaternions (and other hypercomplex systems) operations be reflected in the brain? e.g. perhaps rotation is performed by repeated small, controlled rotational increments. (Research shows that task time is correlated with angle size – amount of rotation.)

Some Topics:

- Is math innate or invented? The brain can compute geometric functions.
  - Computations by the brain (geometric patterns have been induced through psychedelic drugs by Jack Cowan, University of Chicago)
- Animal navigation – emotion and effort; plus location and direction
  - Analogy: Thought trajectory (analog to melody)
  - Quaternion form: S + V (scalar plus vector)
- What promise does quaternions and geometric algebra seem to offer research on the cognitive brain:
  - Geometric generalization facility – solve problems in 4D; return answers in 3D (Hamilton)
  - Interior Selves management facility in Working Memory (Ben Goertzel)

Musical Forms and Hypercomplex Numbers

- Melodies are musical forms in a tonal space.
  - Melodies are geometric shapes reflecting paths taken while traversing a tonal attraction space. Stronger attractions come from shorter tonal distances, measured in harmonic steps of separation of two notes, based on overtone series.
- Some composers have used quaternion, hypercomplex, and projective geometry relationships to create their compositions.
  - Algebra, including quaternions: Gerald Bolzano, Guerino Mazzola
  - Projective geometry: David Lewin
- Coding and interpreting the logistics of movement (Kevin Behan and Mike Mair) – rotation, etc.
- Music is a Simple System – few elements, powerful results, window on cognition
- We can consider music to be the first Virtual Reality (VR) environment experienced by human civilization

Subsection:
4D in Music Cognition and Culture

Fourth Dimension – Math and Culture

Painting (1979): Search for the Fourth Dimension Salvador Dali

В поисках четвертого измерения, 1979. Холст, масло.

Fourth Dimension – Math and Culture

- 1788 – Lagrange, viewed mechanics as a 4D system in Euclidean spacetime
- 1823 – Mobius, showed that in 4D you could rotate a 3D object onto its mirror-image
- 1840 – Grassmann, investigated n-dimensional geometries
- 1843 – Hamilton, invented quaternions, a 4D operational space for rotations and other transformations such as symmetry and scale
- 1853 – Schlafli, developed many polytopes (higher-D polyhedrons) in higher dimensions
- 1880 – Charles Hinton, first to treat the possibility of a 4D physical reality
- 1884 – Edwin Abbott Abbott, Flatland: A Romance in Many Dimensions
- 1905 – Rudolf Steiner, Berlin lecture on the Fourth Dimension
- 1908 – Hermann Minkowski, invented non-Euclidean 4D spacetime; this was used by Einstein
- 1979 – Salvador Dali, Painting: *Search for the Fourth Dimension*
- 2009 – Mike Ambinder, “Human four-dimensional spatial intuition in virtual reality”

Fourth Dimension – Cognition & Neuroscience

Human cognition has an inherent capacity to engage in 4D multisensory processing. This is reflected in the research of:

- Arnold Trehub – autaptic cells (discussed earlier)
- Krumhansl & Kessler -- 4D Perceived Space of Musical Key Distances
- Mike Ambinder – many people can make judgments about lines and angles in a 4D space
- Mike Mair
- Terry Marks-Tarlow
- Ben Goertzel
- Gilles Baroin

Music Cognition – Krumhansl & Kessler (1982)
Derived 4D Perceived Space of Musical Key Distances

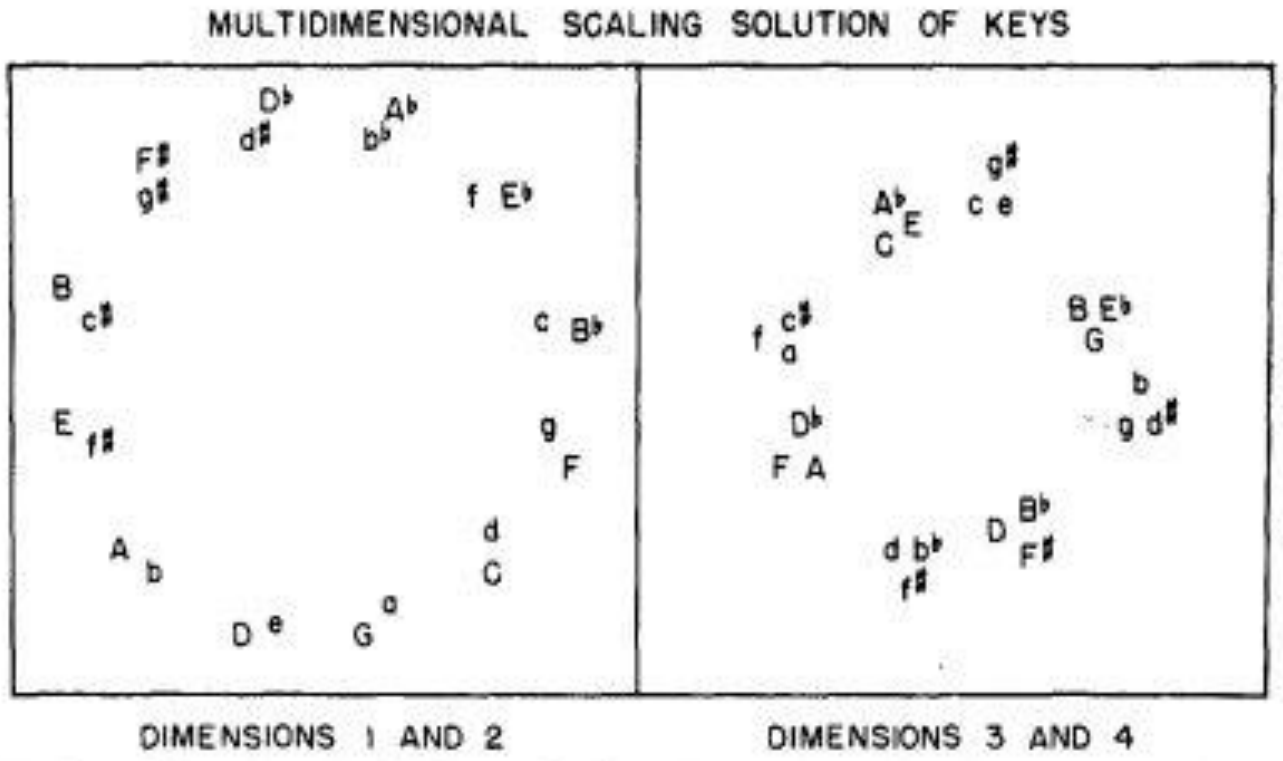


Figure 4. The four-dimensional multidimensional scaling solution of the intercorrelations between the 24 major and minor key profiles (stress = .017). (The projection of the solution onto the first two dimensions is shown on the left. In this projection the circle of fifths for major and minor keys was obtained. The projection onto the last two dimensions is shown on the right. Major and minor keys separated by an interval of a major third were represented as single points in the solution. Each major key was located next to its parallel minor key on one side and its relative minor key on the other. Similarly, each minor key was flanked by its parallel and relative major keys.)

Music Harmony Modeling – Gilles Baroin (2011) Via Unit Hypersphere Quaternions

- Performs a 4D trajectory of musical notes
- Dissertation: Applications of graph theory to musical objects: Modeling, visualization in hyperspace. (University of Toulouse)

- DEMO:
  ACT 5 FOUR DIMENSIONS : THE PLANET-4D PITCH AND CHORDAL SPACE

  We now visualize the pitch space in a true four-dimensional space, by projecting it into our 3D space and letting it rotate around two 4D Axes. The same rotating ball that symbolizes the current position, never moves while the model rotates. Thanks to this technique, the model appears to be deforming within a 3D sphere. That reinforces the feelings of symmetry for the spectator.
- http://youtu.be/MGCIPZyaiuw

Fourth Dimension – Cognition & Neuroscience

- 2009 – Mike Ambinder, Human four-dimensional spatial intuition in virtual reality.

‘Research using virtual reality finds that humans in spite of living in a three-dimensional world can without special practice make spatial judgments based on the length of, and angle between, line segments embedded in four-dimensional space.[12] ’

‘The researchers noted that “the participants in our study had minimal practice in these tasks, and it remains an open question whether it is possible to obtain more sustainable, definitive, and richer 4D representations with increased perceptual experience in 4D virtual environments."[12] ‘ Wikipedia

Ambinder M. S., et al (2009). Human four-dimensional spatial intuition in virtual reality. Psychonomics Bulletin & Review, 16, 5, 818-823
http://link.springer.com/article/10.3758%2FPBR.16.5.818

Music Cognition – Are Melodies 3D or Perhaps 4D? Some Suggestive Evidence

- Musical Key systems are 4D (perceived distances between keys).
  - Perhaps there is a parallelism in dimension between keys and melody via the harmony generating system.
  - At least some of the strictly rotational transformations of melody (non-reversal transformations) in Marina Korsakova-Kreyn's experiment involved key changes, an activity involving re-orientation to a 4D system.
- Melodies are complex and integrated, reflecting the effects of many tonal attraction elements.
- In Mike Mair's nature-of-text research, the melody attribute of text is characterized as 4D, and is described as the **trajectory** of the text.
  - This parallels Panksepp/Behan's interpretation of emotions as guidelines for remembering how to perform a life-essential traverse or journey.
  - "The melody of the text" includes movement such as gestures, ballistics, dance, and oral-facial movements.

Music Cognition – Are Melodies 3D or Perhaps 4D? The Melody of the Text (Mike Mair)

- "Even though the speech trajectories capture virtual world models rather than actual objects on four-dimensional trajectories (like a prey animal moving in the environment), I suggest that the trajectory of speech with movement [*gesture, including ballistic and oral-facial*] is non-verbal, the product of the core brain forming the core to the speech act. The 'point' is the point. A growth point is defined as the 'initial form of thinking out of which speech-gesture organization emerges'. (McNeill) It might also be called the 'projection point'.

- The core brain mechanisms underlying human natural story telling can now be glimpsed. Damasio's core brain text generator in action describes the nonverbal internal structure of gesturing behaviour in speech with movement. It may have functioned projectively on 4D-space time for probably billions of years. Additional control of outcomes is achieved by adding more dimensions or variables to the modeling process, up to our present limit of 7+/-2."

- *Mike Mair, The Melody of the Text – Revisited (c. 2002-2014).*

General Cognition –4D Quaternion Higher-Level Map? The Melody of the Text (Terry Marks-Tarlow)

- “Meanwhile, I also have some thoughts about how to access related ideas neurobiologically. The key brain structure is the hippocampus, historically viewed as the seat of episodic memory, but more recently recognized as the seat of imagination and mental time travel forward as well. In the rat, the hippocampus has been studied as the seat of spatial navigation.
- A guy named Buzsaki has identified 1- and 2-dimensional maps formed by individual place cells. 1-D maps are formed by touch as the rat moves in straight lines (like dead reckoning of sailors). 2-D maps are formed when the rat explores a single point in space from the perspective of many intersecting lines. Once this occurs, the rat is able to calibrate internal sensory motor systems with external features, such that it becomes oriented in physical space and no longer needs to keep track from the inside in order to navigate. Instead, the rat can use outside information, like the sight of a familiar water dish, to navigate around. Buzsaki makes the link from rats exploring physical space to humans encoding episodic memory.

  I believe part of the jump here involves the use of higher dimensional 3-D and 4-D maps in much the same way: a single episode that is remembered concretely involves 3-D grids. Multiple episodes that explore the same territory from a variety of perspectives move onto higher dimensional spaces that allow greater abstraction by removing the event from its concrete context. This may be the basis for semantic memory (baby goes from understanding a single animal as "cat" to "cat" as a generic idea applying to multiple cats; in social space, multiple encounters might get abstracted in higher dimensional space to provide heuristics about how to engage socially or live one's life).”
- Terry Marks-Tarlow*, personal email (February 7, 2015).*

Ben Goertzel – Memory and Mirrorhouses

- **Abstract**. **Recent psychological research suggests that the individual human mind may be effectively modeled as involving a group of interacting social actors: both various subselves representing coherent aspects of personality; and virtual actors embodying "internalizations of others."**
- Recent neuroscience research suggests the further hypothesis that these internal actors may in many cases be neurologically associated with collections of mirror neurons.
- Taking up this theme, we study the mathematical and conceptual structure of sets of inter-observing actors, noting that **this structure is mathematically isomorphic to the structure of physical entities called "mirrorhouses."**

Ben Goertzel – Memory and Mirrorhouses

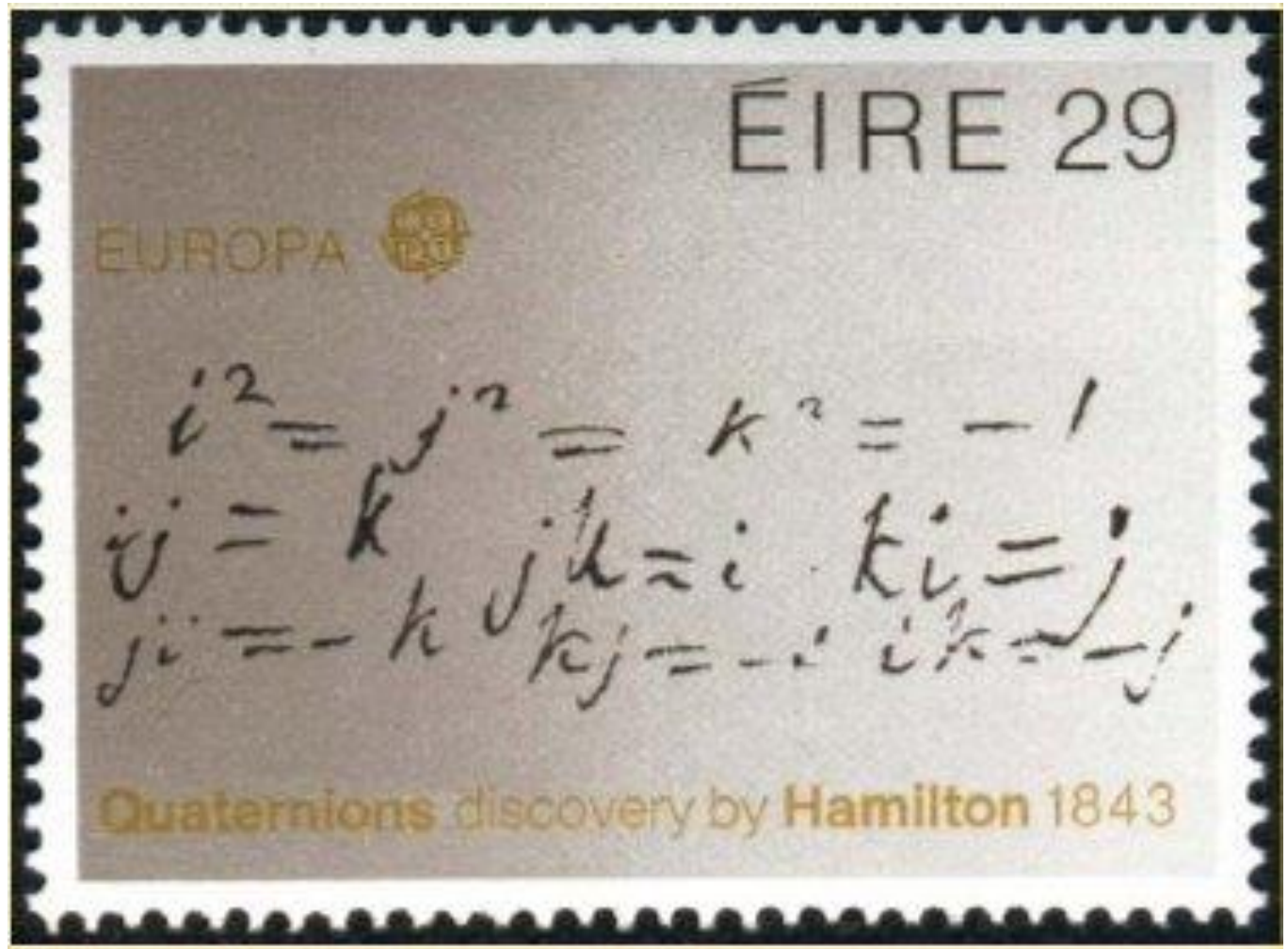
- Mirrorhouses are naturally modeled in terms of abstract algebras such as quaternions and octonions (which also play a central role in physics), **which leads to the conclusion that the presence within a single human mind of multiple inter-observing actors naturally gives rise to a mirrorhouse-type cognitive structure and hence to a quaternionic and octonionic algebraic structure as a significant aspect of human intelligence.**
- Similar conclusions would apply to nonhuman intelligences such as AI's, we suggest, so long as these intelligences included empathic social modeling (and/or other cognitive dynamics leading to the creation of simultaneously active subselves or other internal autonomous actors) as a significant component.

Closing Quotes – Value of Generalization Using Quaternions

- One of the most important ways development takes place in mathematics is via a process of generalization. On the basis of a recent characterization of this process we propose a principle that **generalizations of mathematical structures that are already part of successful theories serve as good guides** for the development of new physical theories.
- The principle is a more formal presentation and extension of a position stated earlier in this century by Dirac.
- **Quaternions form an excellent example of such a generalization and we consider a number of ways in which their use in physical theories illustrates this principle.**
  (*Ronald Anderson, 1992*)

Closing Quotes – Fractals, Betweenness, and Quaternions

- Everywhere they arise, fractals occupy the boundary zone between dynamic, open processes in nature. This quality of betweenness is illuminated by a technical understanding of fractal dimensionality. Since imaginary numbers model hidden dimensionality, in the case of fractals, this consists of infinite expanses, or imaginary frontiers that lurk in the spaces between ordinary, Euclidean dimensions.
- Clouds are zero-dimensional points that occupy three-dimensional space, coastlines one-dimensional lines that occupy two-dimensional planes, and mountains two-dimensional surfaces draping a three-dimensional world. Quaternions are products of the hypercomplex plane consisting of one real and three imaginary axes.
- If imaginary numbers do relate to abstract processes in consciousness, and more specifically to the **fuzzy zone between body and mind**, then because they are three-dimensional shadows of four-dimensional space, **quaternions may provide some clues as to the internal landscape of higher dimensional thought."** *Terry Mark-Tarlow, Semiotic Seams (2004)*


ÉIRE 29
EUROPA
$i^2 = j^2 = k^2 = -1$
$ij = k \quad jk = i \quad ki = j$
$ji = -k \quad kj = -i \quad ik = -j$
Quaternions discovery by Hamilton 1843

END

SUPPLEMENTARY SLIDES

Conjecture: Three Levels or Stages (Frequency, Algebraic, Geometric) of Brain Sensory Processing Strategy

It is my conjecture that the brain, using correlates of algebraic and geometric principles, creates information at three levels of generality. Each level is built on top of the preceding level.

1. **Frequency Detection Level** – sensory frequency information is detected and isolated by attention. (*example: Gustav Herdan (1962) – posited word frequency indexing by brain*)
2. **Analyzing/Structuring Level** -- A set of algebraic polarities are superimposed on the frequency information (e.g. by LGN of the thalamus) – example: key color contrasts of red/green and blue/yellow are applied to light wavelength information, creating a multidimensional system from a single-dimensional system. *Tool example: logic polarities in INRC group.*
3. **Integrating/ConnectingLevel** – completion of the system built by the structuring level. *Tool example: octonions, projective geometry, quaternions in color sphere. Perception example: Circular connection of ends of linear spectra of wavelength – red with blue via violet.*

Vectors and Matrices

SELECTED TIMELINE EVENTS – Matrices (Source: Wikipedia and O. Knill)

200 BC	**Han dynasty: coefficients are written on a counting board.**
1801	**Gauss first introduces [his own treatment of] determinants [they have been around for over 100 years].**
1826	**Cauchy uses term "tableau" for a matrix.**
1844	**Grassmann: geometry in n dimensions (50 years ahead of its epoch [p. 204-205]).**
1850	**Sylvester first use of term "matrix" (matrice=pregnant animal in old French or matrix=womb in Latin as it generates determinants).**
1858	**Cayley matrix algebra but still in 3 dimensions.**
Early 20th Century	**In the early 20th century, matrices attained a central role in linear algebra.[103] partially due to their use in classification of the hypercomplex number systems of the previous century.**

Vector History Timeline

SELECTED TIMELINE EVENTS – Vectors (Source: Wikipedia: Josiah Willard Gibbs)

1880-1884	**Gibbs develops and distributes vector analysis lecture notes privately at Yale.**
1888	**Giuseppe Peano (1858-1932) develops axioms of abstract vector space.**
1892	**Heaviside is formulating his own version of vectorial analysis, and is in communication with Gibbs, giving advice.**
Early 1890s	**Gibbs has a controversy with Peter Guthrie Tait and others [quaternionists] in the pages of *Nature*.**
1901	**Gibbs' lecture notes were adapted by Edwin Bidwell Wilson into a published textbook, *Vector Analysis*, that helped to popularize the "del" notation that is widely used today.**
1910	**The mathematical research field and university instruction have switched over from quaternion tools to vector tools.**

The Engines of Thought: Jean Piaget and the Usefulness of Quaternions

(3) Finally, it leads to the attribution of a central role to ordered pairs, in terms of the notions of operation, class, relation and function.

Here we think it is interesting to point out that Peirce was no doubt the first to draw attention to the fundamental role of ordered pairs, which he in fact called 'elementary relations'.[27] He showed, in particular, that if we consider four dyadic relations such as 'colleague of', 'teacher of', 'student of', and 'classmate of', only certain compositions are possible. This led him to the idea, which we consider fundamental to the study of the genesis of intelligence, of an operation which is not everywhere defined. Actually, this is one of the central ideas of the structure of a hypergroup, as introduced by Menger. Peirce, basing himself on the studies conducted by his father on quaternions, remarked with surprise on the analogy between the table of composition of these four relations and that given by B. Peirce for the quaternions.

Jean Piaget, The Epistemology and Psychology of Functions (1968, 1977)

Controversies – Quaternion Advocates versus Vector Advocates

- Quaternion Advocates: Peter Tait, Knott, MCauley
- Vector Advocates: Gibbs, Heaviside
- Independent View: Cayley – quaternions for pure math, Cartesian coordinates for applied math
- Grand Debate: 1891-1894, 8 journals, 12 scientists, 36 articles.
  Gibbs called it “a struggle for existence” – a Battle of Gettysburg.
  (Wilson’s 1901 textbook, expanding Gibbs’ classroom notes, later decided it).
- Issues
  - Notation and ease of use
  - Familiarity
  - Negative squared quantities
  - Naturalness and closeness to geometric substance
  - Appropriateness for Mathematical Physics and Electromagnetism

Controversies – L.L. Whyte and Dimensionless Approach

“Many workers have considered the relation of quaternions to special relativity and to relativistic quantum theory. If a quaternion is defined, following Hamilton's first method, as a dimensionless quotient of two vectors (lines possessing length, orientation, and sense), **the introduction of quaternions may be regarded as a step towards a dimensionless theory.**

We can interpret Tait's cry,' Repent Cartesian sins and embrace the true faith of quaternions ! ' as meaning 'Drop lengths and substitute angles ! ' Kilmister ' has shown that Eddington's formulation of Dirac's equations can be simplified by using quaternions, and interpreted as representing the non-metrical properties of an affine space of distant parallelism. Thus Dirac's equations in Kilmister's derivation are independent of metric.”

Whyte, L.L. (1954). A dimensionless physics? *The British Journal for the Philosophy of Science, 5, 17*, 1-17

Music/General Cognition – Other Researchers

MUSIC COGNITION

- Fred Lerdahl – Krumhansl's mentor – Melodic Tension, consonance/dissonance
- Hendrik Purwins –torus, keys and notes, model for investing a note with a degree of attraction
- Elaine Chew – cognitive behavior model is Circle of Fifths cylinder plus performer decision-making space

- NEUROSCIENCE AND MATH APPLIED TO MUSIC
- Gyorgy Buzsaki - *Rhythms of the Brain (2006) – oscillations and synchronization*
- Steven Lehar –geometric algebra reflections, oscillations and cycles, standing waves, consciousness – *The Perceptual Origins of Mathematics*; and "Constructive Aspect of Visual Perception: A Gestalt Field Theory Principle of Visual Reification Suggests a Phase Conjugate Mirror Principle of Perceptual Computation."

Rotations – 4D and Double Rotation

- IMPORTANCE OF PLANES:
  In all dimensional spaces (except 1D), rotation is essentially a planar operation.
  Rotation traces out a circle on a plane, which can be used as a template for a cylinder being rotated in 3D.

- IMPORTANCE OF STATIONARY ELEMENTS:
  In 4D, a plane is rotated.
  The plane orthogonal to it is stationary.
  Note: In 3D the stationary element of a rotation is an axis in space; in 2D it is a point in the plane.

- DOUBLE ROTATION:
  In 4D, a second simultaneous but independent rotation can be performed with the otherwise stationary plane because there are enough degrees of freedom. Also, the two angles of rotation can be different.

Rotations – 4D and Double Rotation

EXAMPLE

- Horizontal rotation plus Vertical and 4th dimension inside-out rotation (simultaneous)
  - Like a swivel-chair rotating on its horizontal axis while the top to bottom (vertical axis) is pulling opposite ends through itself (in a 3D projection)

Applications –Signal Processing and Wavelet Math Are Good Partners, opening the Door to Hypercomplex Analysis

(1) Hypercomplex analysis is used to power many wavelet applications.

(2) Hypercomplex approaches, including quaternions, succeed because they can effectively control the frame of reference to best identify the information in the signal. This is yet another application of their ability to relate to orientation questions.

- **"The connection [of wavelet math] to signal processing is rarely stressed in the math literature. Yet, the flow of ideas between signal processing and wavelet math is a success ..."**

- *Book Reference:*
  Dutkay, D.E. and P.E.T. Jorgensen (2000) in Daniel Alpay (ed) (2006). Wavelets, Multiscale Systems, and Hypercomplex Analysis, page 88.
- *Online reference:*
  books.google.com/books?isbn=3764375884

Intellectual History -- Influencers

Charles Sanders Peirce (1839-1914):

- Invented the philosophy of Pragmatism
- Developed a logic based on mathematics (the opposite of George Boole). As early as 1886 he saw that logical operations could be carried out by electrical switching circuits.
- Founded the field of semiotics (study/theory of signs)
- Contributed to scientific methodology, including statistics
- Did not agree with his father that pure math described the workings of the mind of God, as many of the classic Victorian scientists had done

History – Transformation Concepts in Math

- Quaternions, Mental Rotation, and Holographic/Holonomic Brain- Karl Pribram (1980s) – he emphasized the important role of transformations in brain processing – this was resonant with Felix Klein's emphasis of the primacy of transformation groups in modern geometry.