The Culture of Quaternions

The Phoenix Bird of Mathematics

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http://quaternions.klitzner.org
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(see **Resources** Menu for Bibliography)

The Phoenix Bird

The Engines of Thought: Jean Piaget and the Usefulness of Quaternions

Piaget on the Relationship between Mind, Mathematics, and Physics

Evans: Why do you think that mathematics is so important in the study of the development of knowledge?

Piaget: Because, along with its formal logic, mathematics is the only entirely deductive discipline. Everything in it stems from the subject's activity. It is man-made. What is interesting about physics is the relationship between the subject's activity and reality. What is interesting about mathematics is that it is the totality of what is possible. And of course the totality of what is possible is the subject's own creation. That is, unless one is a Platonist.

From a 1973 interview with Richard Evans (Jean Piaget: The Man and His Ideas)

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1. INTRODUCTION - discovery of *new uses after a long period of neglect*
2. APPLICATIONS – *benefits; and how quaternions operate*
3. HISTORY AND CONTROVERSIES – *perceptions of quaternions*
4. MATH – *nature of quaternions*
5. COGNITION, MUSIC, AND 4D – *potential for new uses of quaternions (short <u>summation</u> of a longer treatment)*

Introduction--
The Word *Quaternion*

- The English word quaternion comes from a Latin word *quaterni* which means grouping things **"four by four."**
- A passage in the New Testament (Acts 12:4) refers to a **Roman Army detachment of four quaternions** – 16 soldiers divided into groups of four, who take turns guarding Peter after his arrest by Herod. So a quaternion was a **squad of four soldiers**.
- **In poetry**, a quaternion is a **poem using a poetry style** in which the theme is divided into four parts. Each part explores the complementary natures of the theme or subject. [*Adapted from Wikipedia*]
- In mathematics, quaternions **are generated from four fundamental elements (1, i, j, k).**
- Each of these four fundamental elements is associated with a unique dimension. **So math quaternions are, by nature, a 4D system.**

The Arc of Dazzling Success and Near-Total Obscurity

Quaternions were created in 1843 by William Hamilton.
Today, few contemporary scientists are familiar with, or have even heard the word, **quaternion**. (Mathematical physics is an exception.) And yet --

- During the 19th Century quaternions became very popular in Great Britain and in many universities in the U.S. (Example: Harvard)
- **Clerk Maxwell** advocated the use of quaternions as an aid to understanding scientific concepts.

However, in the 20th Century (after 1910), quaternions were essentially discarded by most of the math profession when the tools of vector analysis and matrix algebra became sufficiently developed and popularized. A small minority of researchers continued to see their value, among them developmental psychologist Jean Piaget.

- **Ironically, the basic ideas of vector analysis were derived from Hamilton's quaternions.**
- Echoing the **Phoenix Bird** theme, today quaternions have been discovered by a new generation of cutting-edge engineers and scientists in many fields.

Surprising Resurgence and Emerging Use in Biology & Neuroscience

In the 20th Century, especially in the closing decades, quaternions have been applied successfully to every level of nature:

- from aerospace navigation to quantum physics spin.
- from DNA string analysis to explaining development of logic in children.

Quaternion systems do the following well:

- perform rotations
- determine orientation
- shift viewpoint of perception
- filter information
- provide process control

Quotations

- "Quaternions came from Hamilton after his really good work had been done, and though beautifully ingenious, **have been an unmixed evil** to those who touched them in any way, including Clerk Maxwell."

  (*Lord Kelvin, 1892, Letter to Heyward). Quoted by Simon Altmann in Rotations, Quaternions, Double Groups*).

- **"Our results testify that living matter possesses a profound algebraic essence. They show new promising ways to develop algebraic biology."**
  (*Petoukhov, 2012, from his DNA research using quaternion and octonion methods, in* <u>The genetic-code, 8-dimensional hypercomplex numbers and dyadic shifts</u>)

Quotations

“An interest [in] quaternionic numbers essentially increased in last two decades when a new generation of theoreticians started feeling in quaternions deep potential yet undiscovered.“

A.P. Yefremov (2005)

“Quaternions…became a standard topic in higher analysis, and today, they are in use in computer graphics, control theory, signal processing [including filtering], orbital mechanics, etc., mainly for representing **rotations and orientations in 3-[dimensional] space**.”

Waldvogel, Jorg (2008)

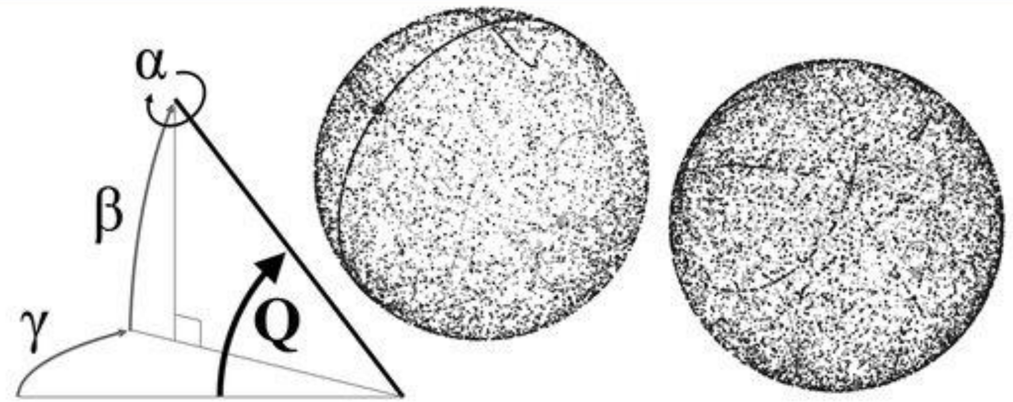
Brief Description of a Quaternion

So what is a quaternion?

- This math relates algebra to 3D space geometry (using 4D tools).
- A quaternion is an algebraic object with a "quantity" part and a "pointing or location" part.
- These are called the scalar and the vector parts, so Q= S + V .
- The S part is real and the V part is made of imaginary number components, i, j, k, -- all three are different: They each behave as different unit directions or independent unit axes.
- So quaternion Q = S + V which = w + (xi + yj +zk), where w, x, y, z are real number coefficients.
- Comments:
  - These quaternion objects occupy a 4D space. It was the first 4D math space in history to be created.
  - Any point (w,x,y,z) in any 4D real number space can be interpreted as a quaternion object, which in turn has meaning as a rotation and sizing operator.

Quaternions and Rotation

Abstract



Much work has been done on algorithms for structure-based drug modeling in silico, and almost all these systems have a core need for three-dimensional geometric models. The manipulation of these models, particularly their transformation from one position to another, is a substantial computational task with design questions of its own. Solid body rotation is an important part of these transformations, and we present here a careful comparison of two established techniques: Euler angles and quaternions. The relative superiority of the quaternion method when applied to molecular docking is demonstrated by practical experiment, as is the crucial importance of proper adjustment calculations in search methods.

4D and Double Rotation

- IMPORTANCE OF PLANES:
  In all dimensional spaces (except 1D), rotation is essentially a planar operation.
  Rotation traces out a circle on a plane, which can be used as a template for a cylinder being rotated in 3D.

- IMPORTANCE OF STATIONARY ELEMENTS:
  In 4D, a plane is rotated.
  The plane orthogonal to it is stationary.
  Note: In 3D the stationary element of a rotation is an axis in space; in 2D it is a point in the plane.

- DOUBLE ROTATION:
  In 4D, a second simultaneous but independent rotation can be performed with the otherwise stationary plane because there are enough degrees of freedom. Also, the two angles of rotation can be different.

Visualization of Movement of Musical Notes– Gilles Baroin (2011) Via Unit Hypersphere Quaternions

- Performs a 4D trajectory of musical notes
- Dissertation: Applications of graph theory to musical objects: Modeling, visualization in hyperspace. (University of Toulouse)

- DEMO:
  ACT 5 FOUR DIMENSIONS : THE PLANET-4D PITCH AND CHORDAL SPACE

  We now visualize the pitch space in a true four-dimensional space, by projecting it into our 3D space and letting it rotate around two 4D Axes. The same rotating ball that symbolizes the current position, never moves while the model rotates. Thanks to this technique, the model appears to be deforming within a 3D sphere. That reinforces the feelings of symmetry for the spectator.
- http://youtu.be/MGCIPZyaiuw

Applications

- BENEFITS -- Examples of Quaternion Application to Problems and Processes
- ARCHETYPE – Spatial Rotation, Orientation, and Alteration of the Frame of Reference

Applications – Partial List

- The list represents a great variety of tasks and interests based on **orientation, filtering, smoothing, and control**:

  - Virtual Reality
  - Real and mental rotation
  - Mathematical Physics problems (e.g. Maxwell Equations, quantum physics)
  - Aerospace – space shuttle pilot software
  - Computer graphics, video games, smooth interpolation
  - **DNA genomic analysis**
  - Bio-logging (animal locomotion orientation)
  - **Music composition**
  - Intellectual development of logic
  - Imbedded schema augmentation in human development
  - **Eye tracking**
  - Supergravity
  - Signal processing and filtering
  - Control Processing and Frame (of Reference) Control
  - Color Face Recognition
  - Quantum Physics (e.g. Dirac and Special Relativity – 2x2 Pauli Spin Matrices)

Applications - Aerospace

Applications – Aerospace Guidance

- Guidance equipment (gyroscopes and accelerometers) and software first compute the location of the vehicle and the orientation required to satisfy mission requirements.
- Navigation software then tracks the vehicle's **actual location and orientation**, allowing the flight controllers to use hardware to transport the space shuttle to the **required location and orientation.** Once the space shuttle is in orbit, the Reaction Control System (RCS) is used for **attitude control**.
- Attitude is the orientation the space shuttle has relative to a frame of reference. The RCS jets control the attitude of the shuttle by affecting rotation around all three axes.
- **Garmin (GPS fame) was an advisor to NASA on determining "Where are we in space?"**
- **Three terms, pitch, yaw, and roll, are used to describe the space shuttle's attitude. Moving the nose up and down is referred to as "pitch," moving the nose left and right is referred to as "yaw," and rotating the nose clockwise or counterclockwise is referred to as "roll" (Figure 1)."**

From: http://www.nasa.gov/pdf/519348main_AP_ST_Phys_RollManeuver.pdf

Applications – Aerospace – Elements of Movement

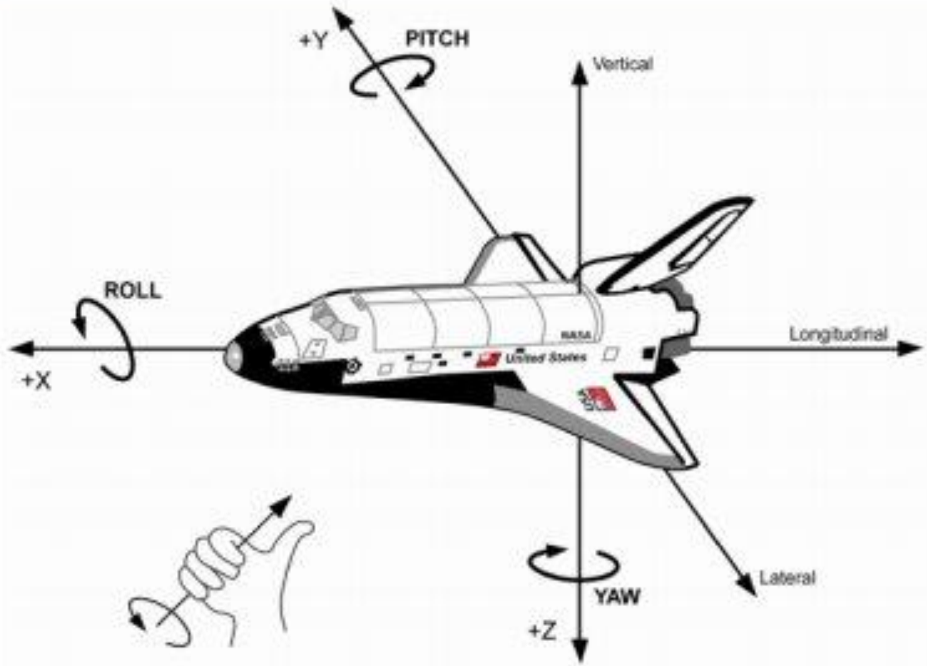


Figure 1: Diagram of the X, Y, and Z body Axes of the space shuttle. Rotation about the X-axis is called Roll, about the Y-axis is called Pitch, and about the Z-axis is called Yaw. The direction of the rotation follows the right hand rule, which states that the thumb of the right hand would be aligned with the positive axis and the direction of the rotation is positive in the direction of the fingers when curling around the axis. The arrows show positive rotation.

The RCS consists of thirty-eight primary jets that produce 3870 Newtons (870 lbs) of thrust each and six vernier jets that produce 107 Newtons (24 lbs) of thrust each. The smaller thrust of vernier jets allow for greater precision of movement. Fourteen primary and two vernier RCS jets are located in the nose of the vehicle and are called forward RCS jets (Figure 2). Twenty-four primary jets and four vernier jets are found on each of the two Orbital Maneuvering System (OMS) pods, located on both sides of the vertical tail and are called aft RCS jets (Figure 3).

Applications – Aerospace Quaternion Advantages

Benefits of Quaternions –

Faster, safer, more compact mathematical expression, easier to identify the angle of rotation – it appears directly in the quaternion.

- When composing several rotations on a computer, rounding errors necessarily accumulate. A quaternion that's slightly off still represents a rotation after being normalised: a matrix that's slightly off may not be orthogonal anymore and is harder to convert back to a proper orthogonal matrix.
- Quaternions also avoid a phenomenon called gimbal lock which can result when, for example in pitch/yaw/roll rotational systems, the pitch is rotated 90° up or down, so that yaw and roll then correspond to the same motion, and a degree of freedom of rotation is lost. In a gimbal-based aerospace inertial navigation system, for instance, this could have disastrous results if the aircraft is in a steep dive or ascent.
  This danger was portrayed in the film, Apollo 13.
- (Wikipedia)

Applications – Aerospace Quaternion Advantages

There are three historical ways to perform a mathematical rotation of a 3D object:

-- orthogonal matrix,

-- Euler angle

-- quaternion

- The representation of a rotation as a quaternion (4 numbers) is more compact than the representation as an orthogonal matrix (9 numbers).
- Furthermore, for a given axis and angle, one can easily construct the corresponding quaternion, and conversely, for a given quaternion one can easily read off the axis and the angle. Both of these are much harder with matrices or Euler angles.

- (Wikipedia)

Applications – Celestial Mechanics

Using quaternions to **regularize** celestial mechanics
(avoiding paths that lead to collisions)

"Quaternions have been found to be the ideal tool for developing and determining the theory of spatial regularization in Celestial Mechanics."

Waldvogel, Jorg (2008). Quaternions for regularizing Celestial Mechanics: The right way. Celestial Mechanics and Dynamical Astronomy, 102: 149-162

Applications – Computer Graphics

In Film Animation, the quaternion method is used by almost all large animation studios today, and is used by small studios as well. There are standard software packages for this purpose.

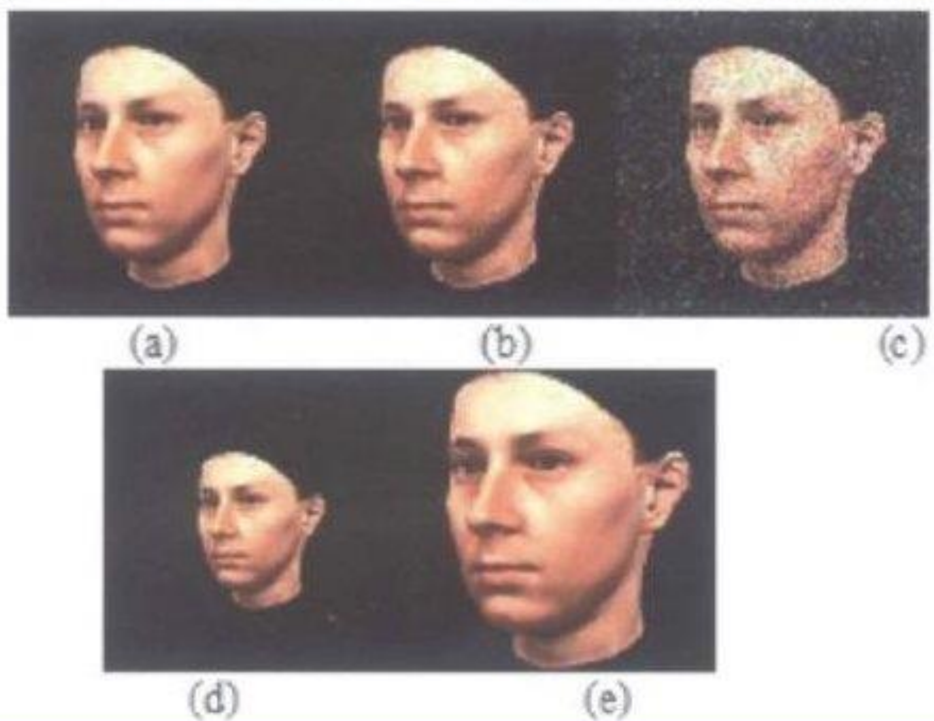
This is because they use a smooth interpolation method drawn from spherical (3D) trigonometry.

- In video games and other applications including animation, one is often interested in "smooth rotations", meaning that the scene should slowly rotate [instead of jumping] in a single step.
- This can be accomplished by choosing a curve such as the spherical linear interpolation [SLERP] in the quaternions, with one endpoint [of the curve] being the identity transformation 1 (or some other initial rotation) and the other being the intended final rotation.
- **This is more problematic with other representations of rotations.** (Wikipedia)

Applications – Color Face Recognition / Pattern Recognition

Quaternion Advantages: Speed, Accuracy (per Wai Kit Wong)

Figure 10.12 An example of a person set (a) original image (b) embedded with mild salt and pepper noise, (c) embedded with heavy salt and pepper noise, (d) shrink (e) dilation.



Applications – Color Face Recognition / Pattern Recognition

Quaternion Advantages: Speed, Accuracy (per Wai Kit Wong)

Table 10.2 Normalized enrollment stage time consumption, normalized classification stage time consumption, and matching accuracy for different color face classification method

Color face classification method	Enrollment stage normalized time consumption (for training all data sets in database)	Classification stage normalized time consumption (for matching 10,000 tested image)	Accuracy (output human names/ID match with the correspondence input images)
Conventional NMF	2.76	1.39	80.18%
BDNMF	3.51	1.55	83.37%
Hypercomplex Gabor filter (Mahalanobis distance classification)	1.36	1.20	86.13%
Quaternion-based fuzzy neural network classifier	1.00	1.00	92.06%

Applications – Color Representation and Image-Signal Processing

Preventing hue distortion

In classical image filtering, we typically have to handle the problem of color space *closure*. Image pixel values have a finite range, typically positive and scaled to the interval [0,1]. If a filtering operation results in a pixel value outside this range, then it must be brought back into the range, ideally without distortion artifacts. One says the range of legitimate pixel values is not closed with respect to the filtering operation. For grayscale images, closure is typically forced by clipping the pixel value at the range boundaries. For color image filters, the color space is a 3D bounded volume, e.g., the RGB color space is typically mapped to a unit edge-length cube with one corner (the black color) at the origin. The color space closure problem becomes more difficult, especially if we wish to avoid visual distortions in the color image output. To reduce these distortions, the origin is moved to the center of the color cube. This means that all pixel values are now vectors pointing away from mid-gray, the center of the cube. Now closure can be enforced by clipping output pixel values to the value where the pixel vector passes through the cube's surface. This three-space clipping eliminates hue distortions by holding the orientation of the pixel vector constant during the clipping operation, i.e. only the length of the vector is altered [SAN 04].

Ell, T., Le Bihan, N., and S. Sangwine (2014). Quaternion Fourier Transforms for Signal and Image Processing. Wiley.

Applications – Signal Processing, Wavelets, and Orientation Change in Hypercomplex Analysis

This is how quaternion abilities in rotation and orientation produce superior results in signal and image processing – just turn the frame of viewing in order to see the signal/image more clearly.

‘Hypercomplex approaches [to signal processing], including using quaternions, succeed because **they can effectively control the frame of reference to best identify the information in the signal.** This is yet another application of the ability of quaternions to process orientation issues.”

- *Book Reference:*
  Dutkay, D.E. and P.E.T. Jorgensen (2000) in Daniel Alpay (ed) (2006). Wavelets, Multiscale Systems, and Hypercomplex Analysis, page 88.
- *Online reference:*
  books.google.com/books?isbn=3764375884

Applications – Bio-logging
Motion Capturing and Analysis

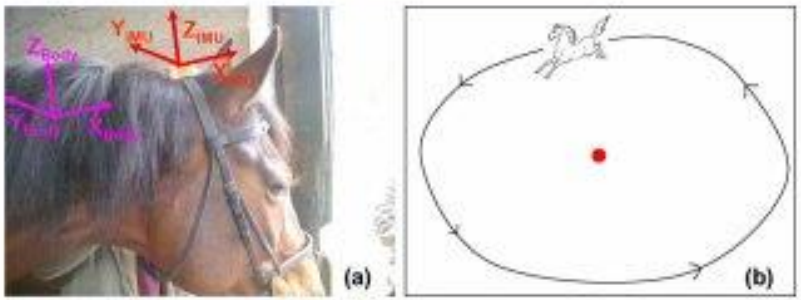


Fig. 12. (a) MTi-G attached to the head of the horse. (b) Schematic diagram of how the horse performed its motion.

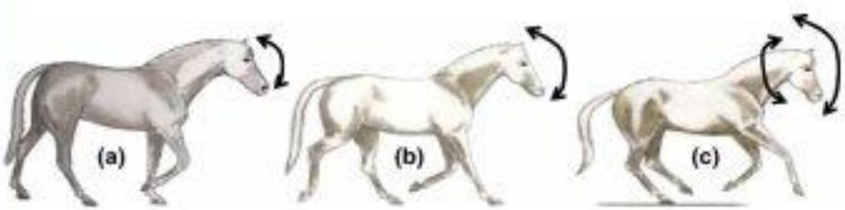


Fig. 13. Gaits of the horse and the movement performed by the head. (a) Walk. (b) Trot. (c) Gallop.

Fig. 9. Subject with the MTi-G attached to the hand.

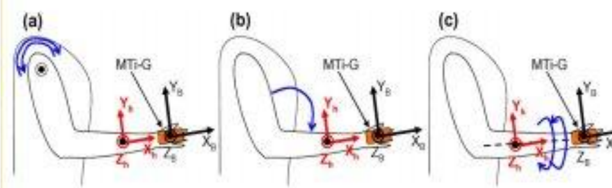


Fig. 10. Exercises performed during the hand motion. (a) Clockwise and anti-clockwise shoulder rotation. (b) Elbow extension. (c) Clockwise and anti-clockwise rotation around the dashed line axis defined along the forearm segment.

Applications – Bio-logging

This study by Hassen Fourati, et al is about body attitude and dynamic body acceleration in sea animals. Using quaternions makes these kinds of studies more practical to carry out.

3D Analysis Gives Better Results Than 2D,
and Quaternions Excel in 3D Motion Analysis

- "**Marine animals** are particularly hard to study during their long foraging trips at sea. However, the need to return to the breeding colony gives us the opportunity to measure these different parameters using bio-logging devices."
- "Note that the use of inertial and magnetic sensors is relatively recent, due to the difficulty to develop **miniaturized technologies** due to **high rate record sampling** (over 10-50 Hz)."
- "**The obvious advantage to this new approach is that we gain access to the third dimension space**, which is a key to a good understanding of the diving strategies observed in these predators…"

Hassen Fourati et al, A quaternion-based Complementary Sliding Mode Observer for attitude estimation: Application in free-ranging animal motions.

Applications – Bio-logging
Energy Expenditure of Animals

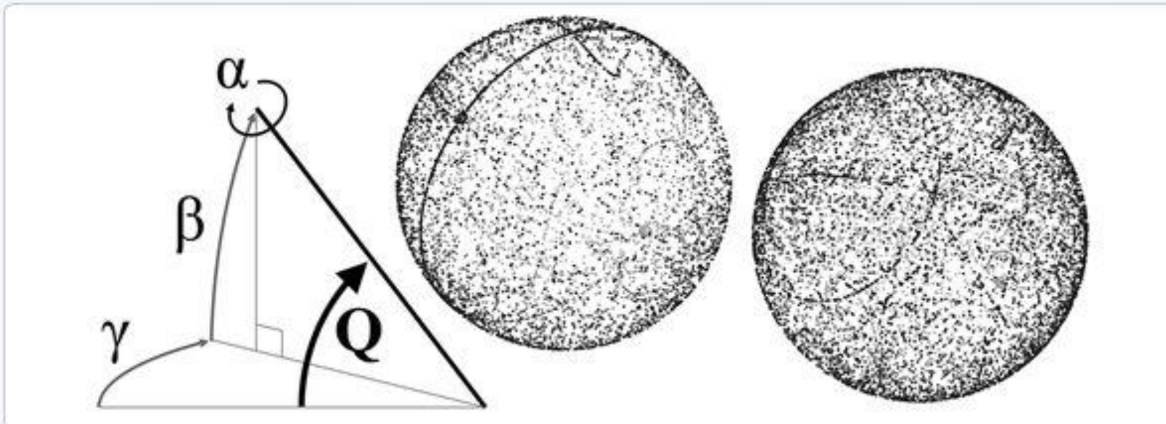
"The proposed approach combines a quaternion-based nonlinear filter with the Levenberg Marquardt Algorithm (LMA). The algorithm has a complementary structure design that exploits measurements from **a three-axis accelerometer, a three-axis magnetometer, and a three-axis gyroscope."**

- **ABSTRACT** This paper addresses the problem of rigid body orientation and dynamic body acceleration (DBA) estimation. **This work is applied in bio-logging, an interdisciplinary research area at the intersection of animal behavior and bioengineering.**

  ….Attitude information is necessary to calculate the animal's DBA [dynamic body acceleration] **in order to evaluate its energy expenditure**.
- Journal Reference:
- Hassen Fourati, Noureddine Manamanni, Lissan Afilal, Yves Handrich (2011). **A Nonlinear Filtering Approach for the Attitude and Dynamic Body Acceleration Estimation Based on Inertial and Magnetic Sensors: Bio-Logging Application. IEEE Sensors Journal, 11,1: 233-244**

Applications – Pharmaceutical Molecules and Receptor Docking

Abstract



Much work has been done on algorithms for structure-based drug modeling in silico, and almost all these systems have a core need for three-dimensional geometric models. The manipulation of these models, particularly their transformation from one position to another, is a substantial computational task with design questions of its own. Solid body rotation is an important part of these transformations, and we present here a careful comparison of two established techniques: Euler angles and quaternions. The relative superiority of the quaternion method when applied to molecular docking is demonstrated by practical experiment, as is the crucial importance of proper adjustment calculations in search methods.

Applications – Pharmaceutical Molecules and Receptor Docking

Article: "Doing a Good Turn: The Use of Quaternions for Rotation in Molecular Docking"

- They use a quaternion analysis of molecule maneuvering and docking in order to understand the dynamics of 3D molecular motion near receptors.
- This parallels quaternion uses in studying 3D orientation in animal motion and space shuttle flight docking.
- "The manipulation of these [3D] models, particularly their transformation from one position to another, is a substantial computational task with design questions of its own."
- "The relative superiority of quaternion methods .... is demonstrated by practical experiment."

- **Skone, Gwyn, Stephen Cameron *, and Irina Voiculescu (2013)** Doing a Good Turn: The Use of Quaternions for Rotation in Molecular Docking. J. Chemical Information and Modelling (ACS), 53(12), 3367-3372 http://pubs.acs.org/doi/abs/10.1021/ci4005139 Oxford research team

Applications – Organic Chemistry

Tetrahedron structure and quaternion relationships

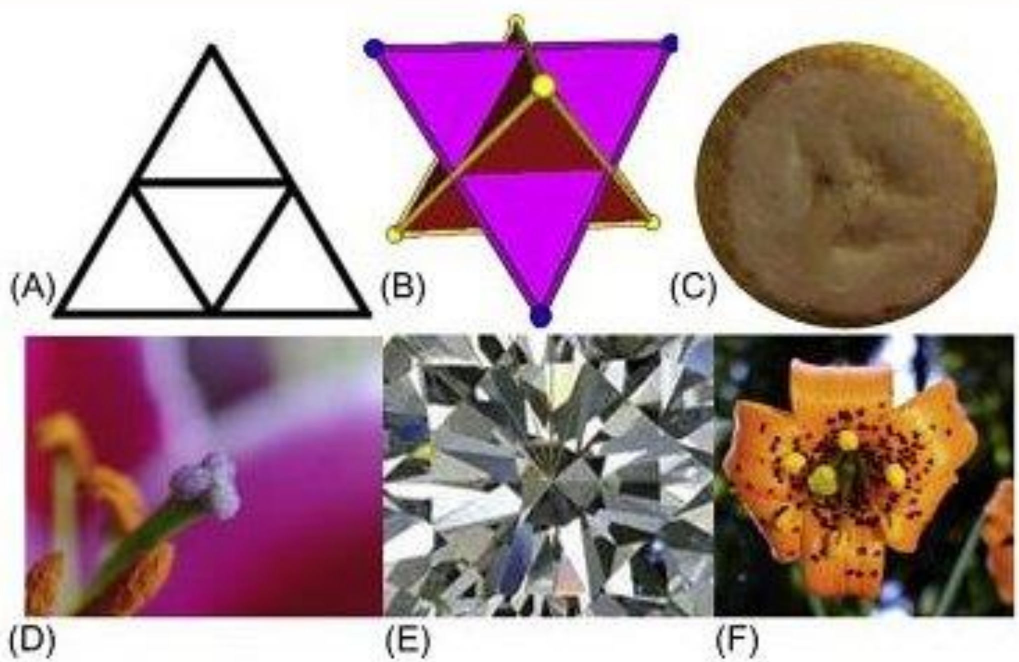


Figure 6.9 Tetrahedral symmetry.

Applications – Organic Chemistry
Methane, Ammonia, and Tetrahedron Structure

Tetrahedron structure and quaternion relationships

- "A leading journal in organic chemistry is called "Tetrahedron" in recognition of the tetrahedral nature of molecular geometry."
- "Found in the covalent bonds of molecules, tetrahedral symmetry forms the methane molecule (CH_4) and the ammonium ion (NH_4^+) where four hydrogen atoms surround a central carbon or nitrogen atom."
- "Italian researchers Capiezzolla and Lattanzi (2006) have put forward **a theory of how chiral tetrahedral molecules can be unitary quaternions**, dealt with under the standard of quaternionic algebra."
- **Chiral tetrahedral molecules can be better understood through experimental predictions offered by the unitary quaternion model.**

Dennis, L., et al (2013), The Mereon Project: Unity, Perspective, and Paradox.
Capozziello, S. and Lattanzi, A. (2006). Geometrical and algebraic approach to central molecular chirality: A chirality index and an Aufbau description of tetrahedral molecules.

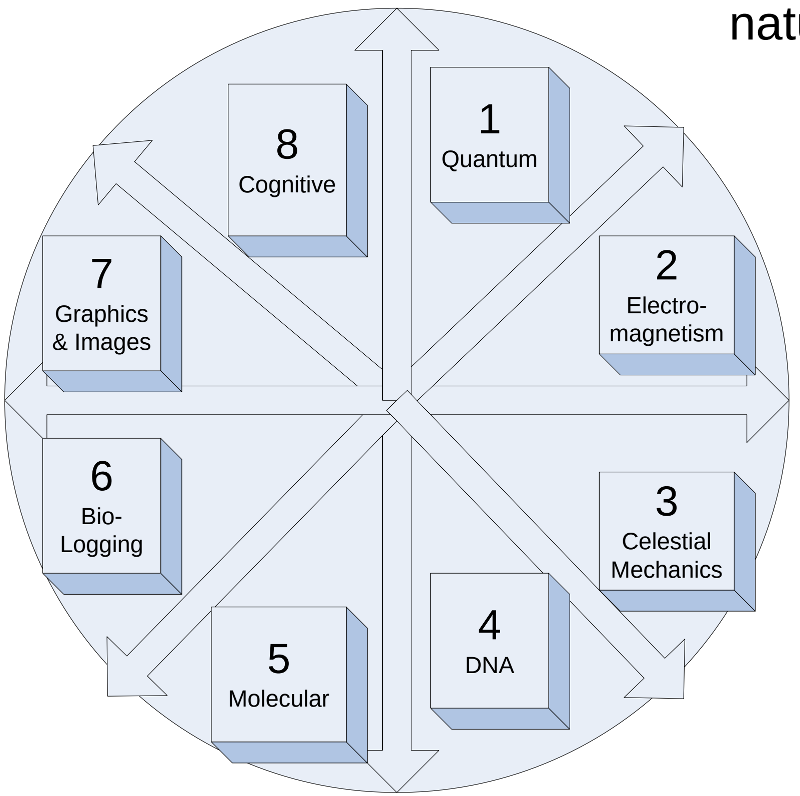
Applications - Quantum Mechanics

- The quaternions are closely related to the various "spin matrices" or "spinors" of quantum mechanics (e.g. the set of four Pauli 2x2 spin matrices).
- Objects related to quaternions arise from the solution of the Dirac equation for the electron. The non-commutativity of the multiplicative product is essential there.

References:

- White, S. (2014). Applications of quaternions. www.zipcon.net
- Finkelstein, Jauch, Schiminovich, and Speiser *Foundations of Quaternion Quantum Mechanics*, J. Math. Phys, **3** (1962) 207-220

Applications – Represent Objects at All Levels of Nature



Is a propensity to quaternion relationships built into nature?

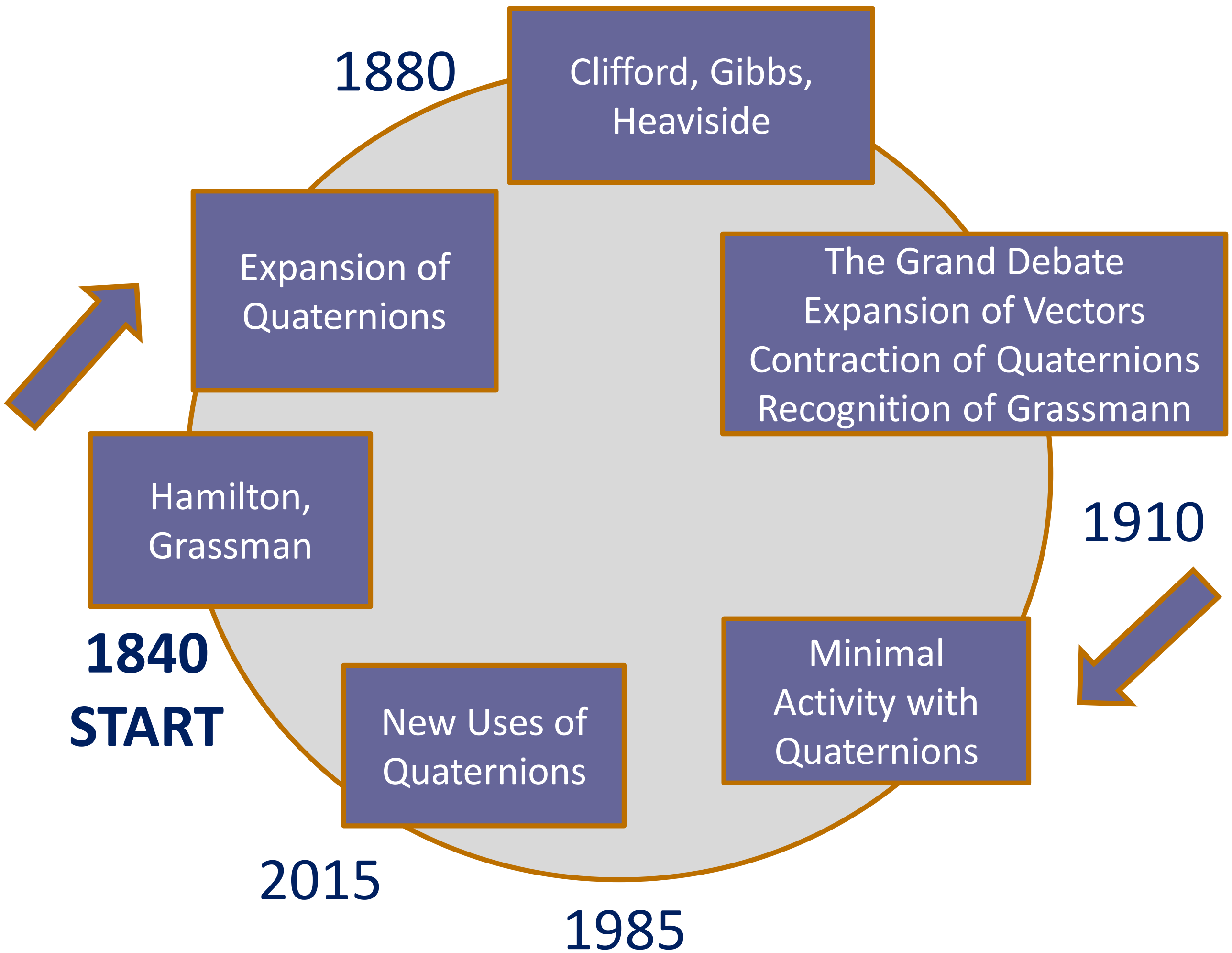
Scale of Nature

- Sub-atomic
- Atomic
- Molecular
- DNA
- Bio-logging
- Cognitive
- Celestial

History

- Phoenix Cycle (diagram)
- Ada Lovelace, Clerk Maxwell
  (Lovelace was the collaborator with Babbage in designing a 19th century proto-computer)
- Benjamin Peirce, Jean Piaget

History Overview – Quaternions vs Vectors



TIME CIRCLE
1840-2015

Historical Overview -- Personalities

Per-iod	Era	Personalities
1	Mid-19^{th} C.	Wm. Hamilton (1843), Robt. Graves (1843), Hermann Grassmann (1832, 1840, 1844), Olinde Rodrigues (1840) Ada Lovelace (1843)
2	2^{nd} half 19^{th} C.	Benjamin Peirce (1870), Charles Sanders Peirce (1882), Peter Tait (1867), Clerk Maxwell (1873), (Josiah) Willard Gibbs (1880-1884), Oliver Heaviside (1893), Wm. Clifford (1879), Felix Klein
3	1^{st} Half 20^{th} C.	Jean Piaget (1918), Wolfgang Pauli (1927), Paul Dirac (1930, 1931), E.T. Whittaker (1904, 1943), L. L. Whyte (1954), Nicolas Tesla, E.B. Wilson (1901)
4	2^{nd} half 20^{th} C.	David Hestenes (1966, 1987), Ken Shoemake (1985), Karl Pribram (1986), John Baez (2001), NASA, Ben Goertzel (2007)
	Historians of Math	Michael Crowe (1967), Daniel Cohen (2007), Simon Altmann (1986)
	Philosophers and Educators of Math	Ronald Anderson (1992), Andrew Hanson (2006), Jack Kuipers (1999), Doug Sweetser (2014, www.quaternions.com)

Ada Lovelace – The First Programmer: Multiple imaginary numbers, geometry, poetry, and imagined machines of musical science

Quote from the author, Renat Qayoom:

The square route of negative numbers captivated her. A negative number, when squared, equals a positive one. What number when squared could equal a negative one? A completely different kind of number: an imaginary number. Its attributes could be explained by plotting a graph showing the method of combination of real with imaginary numbers with an axis representing the imaginary component of the complex number and an axis representing the real numbers:

> This created a new form of 2-dimensional geometry (a geometry that 160 years later would be shown to have as one of its distinctive shapes that icon of chaos theory, the Mandelbaum Set).
> Ada's response to this was to ask what sounded like a simple, technical question but which was in fact a deeply profound one: could you have a third set of numbers in addition to real and imaginary ones which would yield a 3 dimensional geometry - A whole new, previously unexplored mathematical space, in other words? De Morgan had already grappled with this question, and he was defeated by it. It was the Irish mathematician Sir William Rowan Hamilton who came up with the answer a few months after Ada posed the question. He had been struggling with the issue for years, and discovered that you had to go up a further dimension to 4 to come up with a workable solution. He called his new numbers quaternions, and they were to prove extremely useful in understanding the bizarre realm revealed by modern physics. [8]

Ada Lovelace -- continued

Ada suggested that if musical tones and pitches and their key relations could be expressed in a scientific language of operation we might be able to produce musical pieces on a machine. She was uniting poetry and geometry by applying different laws, where 2 parallel lines could meet:

> The Analytical Engine has no pretensions whatever to *originate* any thing. It can do whatever we *know how to order it* to perform. It can follow analysis, but it has no power of *anticipating* any analytical relations or truth. Its province is to assist us I making *available* what we are already acquainted with.[9]

Qayoom, R. (2009). *The Complete Works of Rehan Qayoom (Volume I): Prose 1997-2008*. Lulu.com (P. 38, quaternions and music)

Quaternions and Maxwell (1873)

- Maxwell originally wrote the first two chapters of his electromagnetism equations (20 of them) in a customized quaternion notation for the first two chapters, and the rest in standard coordinate notation.
  The quaternions he used were "pure quaternions," meaning using only the "vector" or spatial part without use of the scalar part. He later revised his work to remove the quaternion notation entirely, since many people were unfamiliar with this notation. But he felt that quaternions were a good aid to thinking geometrically, and led to very simple expressions.
- Heaviside re-wrote the Maxwell Equations in 1893, reducing them from 20 to 4 and using vector notation. This was strongly criticized by some scientists, and was celebrated by many others.

Intellectual History -- Influencers

Pioneer	**Quaternion Theory of Relatives (Relations)**	**Models for Child Development of Logic**	**Octonion Advocate and Developer**	**Octonion Applier to Cognition and AI**
				
Benjamin Peirce (1847)	Charles Sanders Peirce (1872)	Jean Piaget (1915, 1968)	John Baez (2001-present)	Ben Goertzel (1980s-present)

Intellectual History -- Influencers

- **Benjamin Peirce (1809-1870)** worked with quaternions for over 30 years, starting in 1847, only 4 years after they were invented by Hamilton.
- Benjamin Peirce was the chairman of the Math Dept. and professor at Harvard, with interests in celestial mechanics, applications of plane and spherical trigonometry to navigation, number theory and algebra. In mechanics, he helped to establish the effects of the orbit of Neptune in relation to Uranus.
- He developed and expanded quaternions into the very important field of linear algebra.
- He wrote the first textbook on linear algebra during 1870-1880, thereby introducing these ideas to the European continent and stressing the importance of pure (abstract) math, a value taught to him by his colleague, Ralph Waldo Emerson, as described in Equations of God, by Daniel Cohen.
- The book was edited and published posthumously by Peirce's son, Charles Sanders Peirce in 1882. (Note: He created semiotics and pragmatism.)

Intellectual History -- Influencers

Jean Piaget (1896-1980)

- Likely the greatest psychologist of Child Development of the 20th Century
- Was influenced by Charles Sanders Peirce, by revisionist mathematics (Bourbaki group), and by the philosophy of Structuralism. He was a Constructivist
- Quaternions were very useful to parts of his work, in development of logic and in development of new schemata via imbedding rather than substitution
- Wrote a philosophical novel when he was 22 (1918) about the ideas of Henri Bergson
- With Barbel Inhelder, wrote the book The Child's Conception of Space (1956), drawing on abstract math including the child's sequentially emerging understanding of the operation of topology, affine geometry, projective geometry, and Euclidean geometry

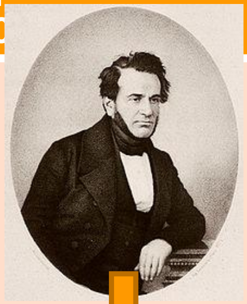
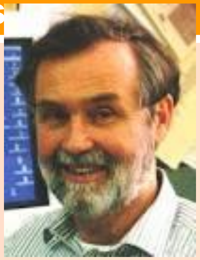
The Engines of Thought
Jean Piaget and the Usefulness of Quaternions

(3) Finally, it leads to the attribution of a central role to ordered pairs, in terms of the notions of operation, class, relation and function.

Here we think it is interesting to point out that Peirce was no doubt the first to draw attention to the fundamental role of ordered pairs, which he in fact called 'elementary relations'.[27] He showed, in particular, that if we consider four dyadic relations such as 'colleague of', 'teacher of', 'student of', and 'classmate of', only certain compositions are possible. This led him to the idea, which we consider fundamental to the study of the genesis of intelligence, of an operation which is not everywhere defined. Actually, this is one of the central ideas of the structure of a hypergroup, as introduced by Menger. Peirce, basing himself on the studies conducted by his father on quaternions, remarked with surprise on the analogy between the table of composition of these four relations and that given by B. Peirce for the quaternions.

Jean Piaget, The Epistemology and Psychology of Functions (1968, 1977)

Quaternion Generalization: Clifford Algebra & Octonion Evolution

William Hamilton Quaternions, 1843	**Hermann Grassmann Geometric Algebra (GA), 1840-1844**	**Olinde Rodrigues Theory of Rotations, (Derived from Euler's 4 squares formula), 1840**	**John T. Graves Octonions, 1843**
			No picture available
William Clifford Clifford Algebra, unified GA, 1878	**David Hestenes Revived/restructured GA, 1950s**	**Simon L. Altmann Quaternions & Rotations, 1986**	**John Baez Octonion applications, 2002**
			

Math

- Tour the Quaternion Neighborhood of Math
- Define the INRC Math "Group" (tessarines)
- Define Quaternions (systems with 3 imaginaries)
- Comment on Rotations and Angles
- Introduce Octonions (systems with 7 imaginaries), and Fano Plane as a Bridge between Quaternion Algebra and Projective Geometry

Math Neighborhood

There are Three Great Branches of Mathematics

- **Analysis**

  (calculus; limit processes)

- **Algebra**

  (combining elements such as numbers; performing symbol operations; arithmetic; solving equations)

- **Geometry**

  (roles and relationships of lines and points, reflection and rotation, spatial representation and processing, concepts such as inside-outside, reversal, and intersection)

Math Neighborhood

- Examples of Number Systems –
  - Natural numbers (including 0)
  - Integers (including negative numbers)
  - Rational numbers
  - Real numbers
  - Complex numbers
- All development is a matter of equilibration. The ultimate goal, equilibration, is an ideal never actually attained. The parts of a system will come into a relative state of balance, be thrown into a disequilibrium again by conflicts with the outside world, and then achieve a higher state of equilibrium.

  Kenneth Kaye explaining Piaget's philosophical novel, Recherche (Quest), 1918.

Extended Math Neighborhood

Hierarchical list– each number system is imbedded in the next.

- Natural numbers
- Integers
- Rational numbers
- Real numbers
- Complex numbers

Hypercomplex Numbers (multiple imaginaries):

- Quaternion numbers
- Octonion numbers
- Geometric Algebra*
- Clifford Algebra systems

*Geometric algebra is a Clifford algebra of a finite-dim. vector space over the field of real numbers endowed with a quadratic form

Algebraic Math Neighborhood

- Some Categories of Algebraic Systems –
  - Groups – one operation, having an identity element and inverses for all elements (making division possible), and closure for operations
  - Rings – 2 operations, having an identity element for each, and with closure
  - Fields – commutative ring, having unique inverses defined for all but the zero element
  - Algebras – ring with dot-product multiplication

- A Powerful Type of Algebra: **The Normed Division Algebra.**
  - There are only four of them.
  - They are nested inside of each other:
    - -- Real (1D)
    - -- Complex (2D)
    - -- Quaternions (4D)
    - -- Octonions (8D)

Math Neighborhood – A Special Hypercomplex Group – The INRC Group

INRC group
(4 elements)

Other names:

- Tessarines (James Cockle, 1848)
- Klein 4-group

i

1

k j

Piaget and the INRC Group (Tessarines): Operations at the Foundation of Logical Capacities

Jean Piaget (1896-1980) [*from webpage of Alessio Moretti,* http://alessiomoretti.perso.sfr.fr/NOTPiaget.html]

‘The Swiss psychologist Jean Piaget, one of the leading figures of "structuralism", on top of his studies on the evolutionary construction of child cognition has proposed a model of the "logical capacities". ‘
‘This is a set of 4 mental operations [on propositions], mutually related by composition laws constituting a mathematical structure of a group, namely a particular decoration [application] of the "Klein 4 group"(because of the 4 operations constituting it), called by Piaget an "INRC group".’

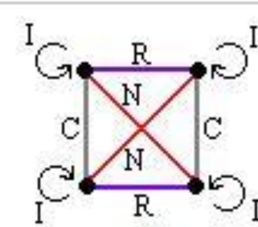
I = identity
N = inversion (negation)
R = reciprocation
C = correlation

basic mental logical operations of young humans

N = RC
R = NC
C = NR
I = NRC

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

Piaget's composition laws making the INRC structure a group



the strange but elegant square structure of the INRC group

Definition of the Unit Quaternion Group

COMPARISON: Quaternions (8 objects) are cousins to the INRC group (4 objects).

- The INRC group Elements: 1, i, j, k (identity element and three elements representing geometric axes)
  - Rules of Combining:
    - **$i^2=j^2=k^2 = 1$,**
    - i times j=k, (NxR=C) -- negating and reciprocating proposition
    - Piaget showed that kids develop understanding of these relationships between logical operations
- Quaternion Group: The above elements <u>plus</u> their negatives
    - **$i^2=j^2=k^2 = -1$, -- we have three <u>different</u> square roots of minus one!**
    - i j = k,
    - ji = -k (non-commutative multiplication – order of elements matters)

Definition of the Quaternion Algebra Space
(By Application of Linear Algebra)

- Now we will create a full quaternion algebra space, not just the group of axis-defined elements.
- These are formed out of linear combinations of the quaternion group elements 1, i, j, k, using real-number coefficients:

$$A + Bi + Cj + Dk$$

Example, substituting coefficients (17, 3, 10, -2):

$17 + 3\,i + 10\,j - 2\,k$ is a quaternion algebra element.
Note: It represents an actual rotation.

- In this space, the elements 1, i, j, k are called <u>basis elements</u> (or simply a "basis") that generate the space of linear combinations.

3D & 4D Rotation Computations

- 4D rotation formula:

  Rotated 4D quaternion = pq, where q is a quaternion to be rotated and p is a quaternion acting as a rotation operator

- 3D rotation formula:

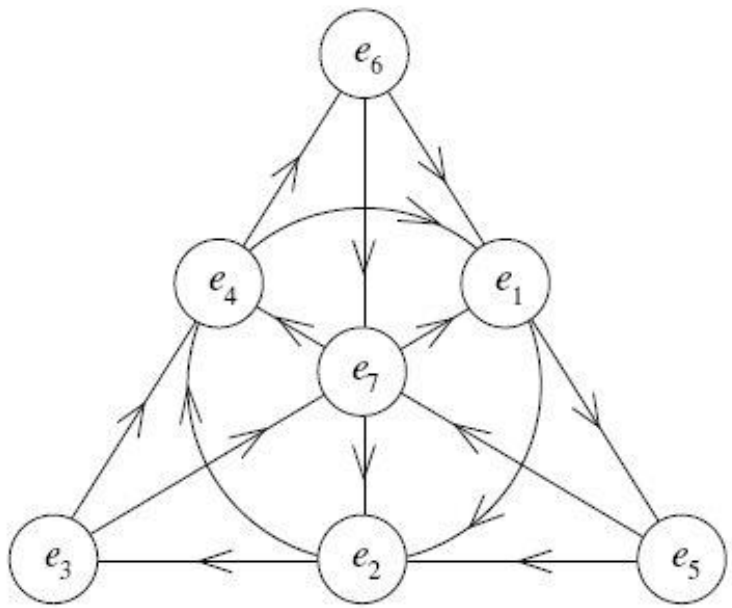
  Rotated 3D vector = pvp^{-1}, where v is a 3D vector to be rotated and p is a quaternion acting as a rotation operator

Octonions

- Invented by William T. Graves in 1843.
- Popularized and developed further by John Baez during the last 15 years (ref. online videos).
- **Octonion elements**: seven independent axes and identity element (1) in an 8-dimensional space.
  - 1, e1, e2, e3, e4, e5, e6, e7 and their negatives.
  - Multiplication is not associative.
- These elements, without the element "1" and the negative elements, form the smallest example of a projective geometry space, the 7-element Fano plane.
- The Fano plane is a GRAND BRIDGE between quaternion algebra and projective geometry!

Fano Plane -- Coding

- Fano Plane coding is a very efficient way of coding items for computer storage



Music, Cognition, and 4D
(Highlights from a Future Talk)

- 3D and 4D Models of Cognition
- 4D Models of Music Cognition
- Quaternions, Fractals, and Thought

“Search for the Fourth Dimension”
Salvador Dali
(Painting, 1979)

Introduction

How do space and music fit together?

Strategic points:

1. Quaternion space is a 4D space. So a quaternion is typically a 4D object.

2. Melody has been described by some researchers as being a 3D, 4D, or even 5D object.

Let us look at some spatial cognitive research findings and conjectures.

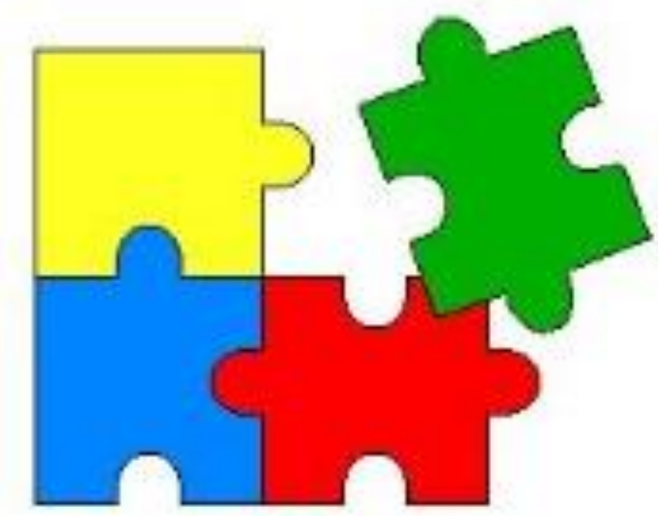
General Cognition and Music Cognition Preliminary Comments

- The foundation of any mathematical model of music is harmony. (The mathematical model of music is based on "tonal attraction." It's a gravity model--gives potential values to each tone for movement toward the tonic note.)
- Music in the brain versus in the air:
  - Acoustics – Sound in the Air
  - Acousmatics – Sound in the Brain – **This one is our interest**.

    Dimensionalities of phenomena in the brain may be different than in acoustics.

3D and 4D Models

3D **General** Cognition Research Models



- 3D Virtual Retinoid Space with Self in Center (Arnold Trehub, 1991, 2005)
- Default 3D Multisensory Space in Parietal Lobe, supported by thalamus (Jerath and Crawford, 2014)
- Supramodal Mental Rotation of Melody and Visual Objects in Parietal Lobe (Marina Korsakova-Kreyn, 2005)

4D **Music** Cognition Research Models

- 4D Distances of Musical Keys From Each Other (Krumhansl & Kessler, 1982)
- Possible 4D Nature of Melodies? (Gilles Baroin, 2011; others)
- 4D/5D Melody of the Text (Mike Mair, 1980)

Fourth Dimension Summary – Cognition & Neuroscience

Human cognition appears to have an inherent capacity to engage in 4D multisensory processing. This is reflected in the research of:

- Arnold Trehub – 4D autaptic cells with short-term memory
- Krumhansl & Kessler -- 4D Perceived Space of Musical Key Distances
- Mike Ambinder – many people can make judgments about lines and angles in a 4D space
- Mike Mair – 4D/5D Melody of the Text experiments
- Terry Marks-Tarlow – 4D Quaternion Spaces in Cognition
- Ben Goertzel – 4D and 8D Mirrorhouse models of internal actors
- Gilles Baroin – 4D Melody – Pitch Trajectory in animation, based on quaternion projection of elements
- Bernd Schmeikal -- 4D basis (tessarines/quaternions) of logic and time-space algebra; Clifford algebra shapes seem to be objects in nature.
- John Gardiner and associates use EEG data as evidence of higher-dimensional fractal signals in the brain, ranging fractionally from 5D to 8D.

Underpinnings of Music: Spatial, Motor, and Affective Pillars

- Cognitive: Navigation and “Where is the Self?” (*Trehub, Marks-Tarlow, Buzsaki, Damasio, Mair)*

- Motor: Locomotion and experience of travel (*Kevin Behan, Daniel Wolpert)*

- Emotion and Body State: Tension/relaxation polarity (*Krumhansl, Panksepp, Korsakova-Kreyn)*

Brain Regions and 3D-4D-8D

Brain Regions: Facts and Relationships

The thalamus connects all brain regions. (Possible 4D-8D conversions)

Perception uses downward flow; imagination uses upward flow. (Dentico)

↑ Occipital lobe

Parietal lobe (3D/4D *assumed by Trehub*)

- Inferior parietal lobe (non-motion) *Trehub, Jerath & Crawford*
- Superior parietal lobe (motion) *Korsakova-Kreyn, Wolpert*

↓ Prefrontal cortex (8D *assumed by Goertzel) Fitch*

Long-term memory formation and spatial navigation.

Prefrontal cortex

↓ Hippocampus *Marks-Tarlow, Buzsaki*

Reciprocal management of declarative/implicit and episodic/explicit knowledge

↑ Parietal lobe

↓ Cerebellum *Weaver, Vandervert*

General Cognition – Trehub Retinoid Model

Here are Arnold Trehub's views on the potential of the retinoid space in the brain to provide 4D capabilities:

> "I'm not knowledgeable enough to respond to your detailed observations about music, but I must point out that all autaptic-cell activity in retinoid space is 4D because autaptic neurons have short-term memory.
>
> This means that there is always some degree of temporal binding of events that are "now" happening and events that happened before "now". The temporal span of such binding probably varies as a function of diffuse activation/arousal.
>
> The temporal envelope of autaptic-cell excitation and decay defines our extended present. This enables us to understand sentences and tunes."
>
> *Personal communication*

ResearchGate.net
Where I Met Arnold Trehub and Many Others

- Free, minimal requirements
- Paper repository
- Lively question discussion groups
- 5 million members
- Heavily international
- Internal messaging is available between members

Quaternions and Carl Jung, and later Jung-Inspired Math-Exploring Researchers

Terry Marks-Tarlow on quaternions:

- "Quaternions are products of the hypercomplex plane consisting of one real and three imaginary axes. If imaginary numbers do relate to abstract processes in consciousness, and more specifically to the fuzzy zone between body and mind, then because they are three-dimensional shadows of four-dimensional space, quaternions may provide some clues as to the internal landscape of higher dimensional thought."

  Semiotic Seams: Fractal Dynamics of Re-Entry (2004)

- This brings us back full circle to Ada Lovelace in 1843:
  How can using multiple imaginary numbers help us represent the relations and operations of aesthetics (and thought) in a scientific way – and re-enact them on a computer?
- I believe there is a great arc of thought connecting the ideas of Ada Lovelace with the newly emerging field of social robotics (the ability of advanced robots to relate comfortably and constructively to people).

End of Presentation